Logical and Process Thresholds for Long-Lifetime Quantum Computation via Surface-Code Quantum Error Correction

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Abstract

Surface–code threshold theorems traditionally assume an abstract local noise model, typically an i.i.d. or weakly correlated Pauli channel on a flat hardware lattice, and establish exponential suppression of logical error below a critical physical error rate [1, 2, 3, 4, 5]. In this setting the surface code functions primarily as a quantum memory architecture, with fault–tolerant computation obtained by supplementing the code with transversal gates and magic–state distillation under the same phenomenological noise assumptions [6, 7]. Such analyses, however, do not natively accommodate curved or slowly drifting chronogeometries, structured vacuum or spectral noise, or process–level equivalence between distinct QEC schedules.

This work develops a triadic–Fourfold fault–tolerance framework for surface–code quantum computation in the IQR/PQS setting of [8]. The starting point is a full fourfold inter–modal correlator $\hat{G}_\Lambda$ and its ideal counterpart $\hat{G}^0_\Lambda$, defined on a QEC diagnostic realized reference configuration and a QEC–realized weave equipped with a Good–clock $C = (\hat{T}, U_t)$. The associated noise kernel $\Delta\hat{G}$ is required to lie in a Triadic–Fourfold Noise Class $\mathcal{N}_{\text{tri-4F}}$ with correlational, vacuum, and spectral sectors satisfying suitable clustering and time–scale separation conditions [9, 10]. From this data the analysis extracts triad–split local error rates $(p_{\text{corr}}, p_{\text{vac}}, p_{\text{spec}})$, triad–resolved leakage parameters $\varepsilon^{(a)}_{\text{ps}}$, and process differentials δ_a as explicit functionals of $\Delta\hat{G}$.

The main Triadic–Fourfold Threshold Theorem shows that the *standard* surface–code family on the emergent Twofold register still exhibits an exponential logical threshold controlled by the correlational error rate p_{corr}, while vacuum and spectral contributions appear only through small, explicitly quantified correction terms $f_{\text{vac}}, f_{\text{spec}}$. At the same time, a triadic process threshold bound ensures that, provided the global process differential δ_{tot} is sufficiently small, all QEC schedules implementing a given logical circuit are holosymmetrically equivalent in the dual–history sense up to a prescribed accuracy. In particular, the noise may arise from a non–Markovian environment, understood here as an environment whose full dynamics is not memoryless or semigroup–like, but whose fast component can still be isolated and treated in a Markovian way for threshold purposes, with the slow, history–carrying components systematically accounted for by the triadic decomposition rather than being ignored. The result is therefore not a new code, but a strictly more general and physically faithful threshold theorem for the usual surface–code architecture: it remains valid in curved, slowly drifting, non–Markovian environments, is formulated directly in terms of field–theoretic correlators, and simultaneously controls both logical correctness and process–level robustness of the computation.

Keywords: fault-tolerant quantum computation, surface-code quantum error correction, stabilizer codes, noise thresholds, correlated and non-Markovian quantum noise, emergent qubits, holosymmetry

1 Introduction

The surface code has emerged as the leading architecture for fault–tolerant quantum computation, owing to its high threshold, locality, and compatibility with two–dimensional hardware [1, 11, 12, 4, 5]. In the standard framework, a family of planar or surface codes $\{\mathcal{C}_L\}$ is defined on a regular lattice of physical qubits, with stabilizer generators of bounded weight and code distance $d_L \sim L$ [13, 14, 15]. Threshold theorems then show that, under a suitable local noise model, one can implement arbitrarily long quantum computations with logical error probabilities that decay exponentially in d_L, provided the physical error rate p lies below a critical value p_* [1, 2, 3, 16, 4, 5]. In this setting, the microscopic details of the environment are abstracted into a small number of channel parameters: typically an i.i.d. Pauli error rate p (possibly with simple spatial/temporal correlations) and occasionally a single leakage rate or an additional "erasure" channel to model level structure beyond the computational subspace [17, 3, 5]. The time variable is treated as an external, flat parameter, and the hardware lattice is assumed rigid and static. These idealisations are extremely effective for hardware–level design and threshold estimation, but they obscure the relationship between fault tolerance and the underlying field–theoretic or gravitational dynamics: slowly drifting backgrounds, curved or relational chronogeometries, and non–Markovian noise with structured correlation functions are either folded into a phenomenological CPTP map or treated as small violations of the standard assumptions, rather than being built into the threshold statement itself [9, 10, 5].

The present work develops an alternative, "triadic–Fourfold" description of surface–code fault tolerance that is explicitly anchored in an *interdependent quantum reality* (IQR) framework [8]. Instead of starting from an *ad hoc* local Pauli channel, the analysis begins with a full fourfold inter–modal correlator $\hat{G}_\Lambda$ and its ideal counterpart $\hat{G}^0_\Lambda$ for the underlying field or many–body system. The difference $\Delta\hat{G} = \hat{G}_\Lambda - \hat{G}^0_\Lambda$ is required to lie in a Triadic–Fourfold Noise Class, in which the noise is decomposed along three physically meaningful axes: fast correlational fluctuations, slowly varying vacuum backgrounds, and spectral (band–structure) effects. On this basis, an emergent Twofold register $\mathcal{H}^{(L)}_{\mathrm{reg}}$ is constructed inside a QEC diagnostic realized reference configuration, and a standard surface–code family $\{\mathcal{C}_L\}$ is defined on that register. The fault–tolerance analysis then proceeds by extracting, from $\Delta\hat{G}$, triad–split error rates $p_{\mathrm{corr}}, p_{\mathrm{vac}}, p_{\mathrm{spec}}$, triad–resolved leakage parameters $\varepsilon^{(a)}_{\mathrm{ps}}$, and process differentials δ_a, together with a triadic QEC time window $t_{\mathrm{corr}} \ll \Delta t_{\mathrm{QEC}} \ll t_{\mathrm{vac}}, t_{\mathrm{spec}}$ that separates fast noise from slowly varying backgrounds [9, 10, 8].

Viewed from this perspective, the surface code is retained as the core stabilizer architecture, but the conditions under which it actually functions as a fault–tolerant quantum computer are rephrased in terms of physically transparent, correlator–level quantities. The main threshold theorem shows that the familiar exponential suppression of logical error is governed by the *correlational* error rate p_{corr} and its associated leakage $\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})}$, while the vacuum and spectral components contribute only through controlled additive corrections $f_{\mathrm{vac}}, f_{\mathrm{spec}}$ that can be made small by appropriate choice of geometry and QEC time window. At the same time, a process–level holosymmetry bound imposes a threshold condition on the global process differential δ_{tot}, ensuring that all reasonable schedules implementing the same logical circuit are physically equivalent in the dual–history sense up to a prescribed accuracy. In this way, a single surface–code family on an emergent register acquires a fault–tolerance guarantee that is *explicitly* compatible with curved and relational chronogeometries, structured non–Markovian environments, and schedule–level variations, while remaining in continuity with the standard threshold picture developed for local noise on stabilizer codes [1, 2, 3, 4, 5, 8].

2 Logical and Process-Level Propositions and Their Valuations in IQR

In this section we fix the ambient setting and the precise language we will use for formulating an IQR-flavoured fault-tolerance threshold theorem [2, 17, 3]. We work throughout in the framework of the Fourfold Dialectic $*$-algebra and its GNS realization, as developed in the Tetralemma–GNS construction and Fourfold correlator framework of [8, Chs. 1–2,4].

2.1 Ambient IQR Setting

Assumption 2.1 (Ambient IQR setting)**.** *We assume the following data [8, Chs. 1–2,4–5]:*

(a) A Fourfold Dialectic $*$-algebra

$$\mathfrak{A}^{(4)}$$

generated (in the sense of the Tetralemma–GNS construction) by:

- *PQS selection operators $\hat{\mathcal{F}}_\alpha(x)$, with mode index $\alpha \in \{0, 1, p, s\}$ and locus $x \in \mathcal{X}_q$, where $\mathcal{X}_q$ is the configuration space of PQS loci (cf. [8, Chs. 1–2]);*
- *transport/connection operators $\mathcal{U}(x \leftarrow x'; \gamma)$, implementing parallel transport along suitable paths $\gamma \subset \mathcal{X}_q$ via the emergent connection [8, Ch. 4];*
- *the Fourfold inter-modal correlator kernel $\hat{G}_\Lambda(\mathfrak{B}_x, \mathfrak{B}_{x'})$, whose components $\hat{G}_\Lambda^{\alpha\beta}(x, x') := \hat{\mathcal{F}}_\alpha^\dagger(x)\,\mathcal{U}(x \leftarrow x'; \gamma)\,\hat{\mathcal{F}}_\beta(x')$ are built from the selectors and the transport [8, Ch. 4].*

(b) A Quantum Holo-State $\omega^{\mathring{\Psi}} : \mathfrak{A}^{(4)} \to \mathbb{C}$ with GNS representation

$$(\pi_\Psi, \mathcal{H}_{\mathrm{em}}, |\Omega_\Psi\rangle),$$

so that each $A \in \mathfrak{A}^{(4)}$ acts as a bounded operator $\pi_\Psi(A) \in \mathcal{B}(\mathcal{H}_{\mathrm{em}})$, and $\omega^{\mathring{\Psi}}(A) = \langle \Omega_\Psi | \pi_\Psi(A) | \Omega_\Psi \rangle$, as in the Holo-State/GNS construction [8, Chs. 1–2].

(c) For each PQS $\mathfrak{B}_x$ there is a modal fibre

$$\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)} \cong \mathbb{C}^4,$$

together with a Fourfold basis $\{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$, obtained via the coincident-limit inner-product construction from the Fourfold correlator and the emergent orthonormalization of the Fourfold frame [8, Ch. 4]. Here the labels $\{0, 1, p, s\}$ correspond to the symmetry, antisymmetry, parasymmetry, and holosymmetry modes of the Fourfold Dialectic.

(d) The Twofold (symmetry–antisymmetry) subspace at x is

$$\mathcal{H}_{(0,1),x} := \operatorname{span}\Big\{|0\rangle_x, |1\rangle_x\Big\} \subset \mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)},$$

which we interpret as the emergent qubit at locus x, in the sense of the Twofold sector and register constructions developed in [8, Chs. 4,9–10].

Definition 2.2 (Emergent register for a PQS patch)**.** Let $\mathcal{P}_L = \{x_1, \ldots, x_{n_L}\} \subset \mathcal{X}_q$ be a finite set of PQS loci, which will support a quantum code of "linear size" L (for instance, a rotated $L \times L$ surface-code patch [18, 19, 20, 13, 16, 21, 1, 22, 4, 15]).

The associated *emergent Twofold register* is the tensor product

$$\mathcal{H}_{\text{reg}}^{(L)} := \bigotimes_{j=1}^{n_L} \mathcal{H}_{(0,1),x_j} \;\cong\; (\mathbb{C}^2)^{\otimes n_L},$$

realizing an n_L-qubit register embedded in the IQR Hilbert bundle, as in the register constructions used for qubit sectors in [8, Chs. 9–10].

In what follows, all encoded logical qubits and surface-code patches will be realized as subspaces of $\mathcal{H}_{\text{reg}}^{(L)}$ for suitable choices of patches $\mathcal{P}_L$ [5, 4, 15, 23, 24].

2.2 Logical Correctness Proposition

We now formalize the logical correctness proposition and its Thesis valuation [2, 17, 3] within the IQR/Tetralemma framework developed in [8, Chs. 1–2], using the emergent Twofold register construction of [8, Chs. 4,9–10].

Definition 2.3 (Surface-code patch and logical subspace)**.** Fix a code size L. A *surface-code patch of size L* consists of [16, 21, 1, 4, 15]:

- a choice of PQS loci $\mathcal{P}_L = \{x_1, \ldots, x_{n_L}\} \subset \mathcal{X}_q$, giving rise to the register Hilbert space $\mathcal{H}_{\text{reg}}^{(L)}$ as in Definition 2.2 and in the Twofold register constructions of [8, Chs. 4,9–10];
- a stabilizer code $\mathcal{C}_L \subset \mathcal{H}_{\text{reg}}^{(L)}$ encoding one logical qubit, so that $\dim \mathcal{C}_L = 2$ [13, 19, 20];
- a stabilizer group $S_L \subset \mathcal{PU}(\mathcal{H}_{\text{reg}}^{(L)})$ (a subgroup of the Pauli group on n_L qubits), with associated code projector

$$P_{\text{SC},L} := \frac{1}{|S_L|} \sum_{g \in S_L} g. \tag{1}$$

We refer to $\mathcal{C}_L$ as the *logical code space* for size L.

Definition 2.4 (Ideal logical channel for one QEC cycle)**.** Let $\mathcal{C}_L$ be a surface-code patch as in Definition 2.3. An *ideal logical channel* for one QEC cycle is a completely positive, trace-preserving (CPTP) map

$$\mathcal{N}^{(L)}_{\text{ideal}} : \mathcal{B}(\mathcal{C}_L) \to \mathcal{B}(\mathcal{C}_L)$$

which, in the simplest case, is the identity channel on the logical qubit. More generally, it may be a fixed logical Clifford or non-Clifford gate acting on the encoded qubit [2, 17, 3, 5].

Definition 2.5 (Physical QEC cycle and induced logical channel)**.** Fix a surface-code patch $\mathcal{C}_L \subset \mathcal{H}^{(L)}_{\text{reg}}$ and consider a *physical QEC cycle* (for a given schedule A) implemented on the full emergent Hilbert space $\mathcal{H}_{\text{em}}$ arising from the Holo-State/GNS construction [8, Chs. 1–2] and the emergent register embedding [8, Chs. 9–10].

1. The physical QEC cycle is modelled as a CPTP map

   $$\mathcal{N}^{(L)}_{A} : \mathcal{S}(\mathcal{H}_{\text{em}}) \to \mathcal{S}(\mathcal{H}_{\text{em}}),$$

   built from:

   - unitary gates acting on PQS loci within $\mathcal{P}_L$ and on nearby ancilla PQS, represented as elements of the Fourfold algebra $\mathfrak{A}^{(4)}$ [8, Chs. 1–2,4];
   - interactions with the rest of the PQS continuum, encoded in the Fourfold correlator $\hat{G}_\Lambda$ and the connection on $\mathcal{X}_q$ [8, Ch. 4];
   - projective or POVM measurements represented as effects in $\mathfrak{A}^{(4)}$ [18, 5, 8].

2. The corresponding *induced logical channel* is a CPTP map

   $$\mathcal{N}^{(L)}_{\text{log}} : \mathcal{S}(\mathcal{C}_L) \to \mathcal{S}(\mathcal{C}_L),$$

   obtained as follows [2, 3, 5, 8]:

   - choose an isometry $V_L : \mathcal{C}_L \to \mathcal{H}_{\text{em}}$ embedding the code space into the emergent arena (using the Twofold sector embedding from [8, Chs. 4,9–10]);
   - given a logical state $\rho_{\text{log}} \in \mathcal{S}(\mathcal{C}_L)$, form the embedded state $\tilde{\rho}_{\text{in}} := V_L \rho_{\text{log}} V_L^\dagger \in \mathcal{S}(\mathcal{H}_{\text{em}})$;
   - apply $\mathcal{N}^{(L)}_{A}$ to obtain $\tilde{\rho}_{\text{out}} := \mathcal{N}^{(L)}_{A}(\tilde{\rho}_{\text{in}})$;
   - project back into the code space and trace out ancilla/environment degrees of freedom:

     $$\mathcal{N}^{(L)}_{\text{log}}(\rho_{\text{log}}) := \mathrm{Tr}_{\text{anc/env}}\Big[P_{\text{SC},L}\, \tilde{\rho}_{\text{out}}\, P_{\text{SC},L}\Big],$$

     with a suitable identification of the resulting density operator as an element of $\mathcal{S}(\mathcal{C}_L)$.

Definition 2.6 (Logical correctness proposition)**.** Let $\mathcal{C}_L$ be a surface-code patch with ideal logical channel $\mathcal{N}_{\text{ideal}}^{(L)}$ and realised logical channel $\mathcal{N}_{\text{log}}^{(L)}$ as above.

We define the *logical correctness proposition* at size L to be the statement

$$\varphi_L^{(L)}: \quad \text{“The encoded logical qubit in patch } \mathcal{C}_L \text{ is preserved through one QEC cycle.”} \tag{2}$$

This is a Tetralemma-type proposition in the sense of the Fourfold Dialectic semantics introduced in [8, Chs. 1–2], specialized here to a yes/no claim about logical preservation.

Definition 2.7 (Worst-case logical error probability)**.** The *worst-case logical error probability* at code size L is

$$p_L^{(L)} \;:=\; \sup_{\rho_{\text{log}} \in \mathcal{S}(\mathcal{C}_L)} \frac{1}{2} \left\| \mathcal{N}_{\text{log}}^{(L)}(\rho_{\text{log}}) - \mathcal{N}_{\text{ideal}}^{(L)}(\rho_{\text{log}}) \right\|_1, \tag{3}$$

where $\|\cdot\|_1$ is the trace norm on trace-class operators [2, 3].

Remark 2.8. The quantity $p_L^{(L)}$ coincides with the worst-case distinguishing advantage between the realised logical channel $\mathcal{N}_{\text{log}}^{(L)}$ and the ideal channel $\mathcal{N}_{\text{ideal}}^{(L)}$ over all input logical states. In the special case where $\mathcal{N}_{\text{ideal}}^{(L)}$ is the identity channel on a single logical qubit, $p_L^{(L)}$ is essentially the worst-case logical failure probability per QEC cycle.

Definition 2.9 (Logical Thesis valuation)**.** The *Thesis valuation* of the logical correctness proposition $\varphi_L^{(L)}$ after one QEC cycle is defined by

$$v_{\text{Th}}(\rho_{\text{out}}; \varphi_L^{(L)}) \;:=\; 1 - p_L^{(L)}. \tag{4}$$

Here ρ_{out} denotes the (family of) output logical states generated by $\mathcal{N}_{\text{log}}^{(L)}$ as ρ_{log} ranges over all inputs; the definition is independent of any particular input. This is the specialization, to a single Tetralemma proposition, of the Thesis valuation component in the Fourfold valuation map [8, Chs. 1–2].

Remark 2.10. The value $v_{\text{Th}}(\rho_{\text{out}}; \varphi_L^{(L)}) = 1$ corresponds to exact equality of the realised and ideal logical channels; any deviation from unity quantifies, in a worst-case sense, the degree to which an optimally chosen logical test could detect a failure of the proposition $\varphi_L^{(L)}$. In the broader Tetralemma semantics, the remaining weight $1 - v_{\text{Th}}$ is distributed among the AntiThesis, Synthesis, and Holothesis valuations, as described in [8, Chs. 1–2], but in the threshold theorem we will only constrain the Thesis component.

2.3 Process Equivalence and Holosymmetry

We next formalize the process-level proposition and its holosymmetric valuation, using a dual-history interferometric construction, within the general framework of fault-tolerant quantum computation and threshold theorems [2, 17, 3, 5], and in direct analogy with the dual-register holosymmetric architecture and Tetralemma valuation semantics developed in [8, Chs. 1–2].

Definition 2.11 (Two QEC histories at size L)**.** For a fixed code size L, let $\mathcal{C}_L \subset \mathcal{H}_{\text{reg}}^{(L)}$ be a surface-code patch with ideal logical channel $\mathcal{N}_{\text{ideal}}^{(L)}$ [16, 21, 1, 4, 15]. A pair of *QEC schedules* (or "histories") at size L consists of two CPTP maps

$$\mathcal{N}_A^{(L)}, \quad \mathcal{N}_B^{(L)} : \mathcal{S}(\mathcal{H}_{\text{em}}) \to \mathcal{S}(\mathcal{H}_{\text{em}}),$$

each built from:

- unitary gates, measurements, and classical feedforward acting on PQS loci in $\mathcal{P}_L$ and on suitable ancillas [18, 20, 13, 5], represented as elements of the Fourfold algebra $\mathfrak{A}^{(4)}$ acting on the emergent Hilbert space [8, Chs. 1–2];
- interactions with the rest of the PQS continuum encoded via the Fourfold correlator and the connection on $\mathcal{X}_q$ [8, Ch. 4];

such that, in the ideal (noise-free) limit, both implement the same logical channel $\mathcal{N}_{\text{ideal}}^{(L)}$ on $\mathcal{C}_L$.

Definition 2.12 (Dual-history holosymmetric interferometer)**.** Let $\mathcal{N}_A^{(L)}$ and $\mathcal{N}_B^{(L)}$ be two QEC schedules at size L as in Definition 2.11. The *dual-history holosymmetric interferometer* is defined by the following steps, as a specialization of the holosymmetric control architecture with a two-level control system and alternative dynamical histories [8, Ch. 2]:

(a) Introduce a reference qubit R, modelled as an additional PQS or ancilla system, and prepare it in the state $|+\rangle_R := \frac{1}{\sqrt{2}}(|0\rangle_R + |1\rangle_R)$, playing the role of the holosymmetric control qubit in [8, Def. Holosymmetric Control System, Ch. 2].

(b) Prepare an encoded logical state $\rho_{\text{log}} \in \mathcal{S}(\mathcal{C}_L)$ and embed it into $\mathcal{H}_{\text{em}}$ via the isometry V_L (cf. the Twofold sector embedding in [8, Chs. 4,9–10]), obtaining an initial joint state

$$\rho_{\text{in}}^{(L)} := |+\rangle\langle+|_R \otimes V_L \rho_{\text{log}} V_L^\dagger \;\in\; \mathcal{S}(\mathcal{H}_{\text{em}}^{(R)} \otimes \mathcal{H}_{\text{em}}).$$

(c) Define a controlled process which applies $\mathcal{N}_A^{(L)}$ when $R = 0$ and $\mathcal{N}_B^{(L)}$ when $R = 1$. At the level of Kraus operators this can be written as

$$\mathcal{N}_{\text{ctrl}}^{(L)}(\cdot) = \sum_i (|0\rangle\langle 0|_R \otimes K_{A,i})(\cdot)(|0\rangle\langle 0|_R \otimes K_{A,i})^\dagger + \sum_j (|1\rangle\langle 1|_R \otimes K_{B,j})(\cdot)(|1\rangle\langle 1|_R \otimes K_{B,j})^\dagger,$$

where $\{K_{A,i}\}_i$ and $\{K_{B,j}\}_j$ are Kraus families for $\mathcal{N}_A^{(L)}$ and $\mathcal{N}_B^{(L)}$, respectively. This is the CPTP analogue, for noisy schedules, of the controlled holosymmetric unitary considered in [8, Ch. 2].

Applying this channel to $\rho_{\text{in}}^{(L)}$ gives the intermediate state $\tilde{\rho}^{(L)} := \mathcal{N}_{\text{ctrl}}^{(L)}\big(\rho_{\text{in}}^{(L)}\big)$.

(d) Apply a Hadamard gate H_R to the reference qubit, obtaining $\rho_{\text{out}}^{(L)} := (H_R \otimes \mathbb{I})\,\tilde{\rho}^{(L)}\,(H_R \otimes \mathbb{I})^\dagger$.

(e) Define the *holosymmetric effect*

$$E_{\mathrm{Holo}} := |0\rangle\langle 0|_R \otimes \mathbb{I}_{\mathrm{sys}}, \tag{5}$$

where $\mathbb{I}_{\mathrm{sys}}$ is the identity on the system degrees of freedom. This is the dual-history specialization of the holosymmetric effect operator used in the general Holosymmetry definition and Born semantics of [8, Ch. 2].

Definition 2.13 (Holosymmetric process valuation)**.** In the setting of Definition 2.12, the *holosymmetric valuation* of the comparison between schedules A and B at size L is the Born probability

$$v_{\mathrm{Holo}}^{(L)} := \mathrm{Tr}\Big(E_{\mathrm{Holo}}\, \rho_{\mathrm{out}}^{(L)}\Big). \tag{6}$$

This is the Thesis component, in the Tetralemma sense, of the holosymmetric effect valuation for the process-level proposition, as in the general Fourfold valuation framework of [8, Chs. 1–2].

Remark 2.14. The quantity $v_{\mathrm{Holo}}^{(L)}$ is the probability that the reference qubit is found in the state $|0\rangle_R$ after the dual-history interference. It quantifies the extent to which the two histories A and B behave as coherent alternatives, rather than as mutually distinguishable processes. In noise models where the difference between A and B is captured by local Pauli channels with single-qubit error-overlap k, one often has a simple form $v_{\mathrm{Holo}}^{(L)} = \frac{1}{2}\Big(1 + k^{m_L}\Big)$, where m_L is the number of PQS involved in the gadget; this is particularly natural in stabilizer and surface-code settings with local Pauli noise [1, 5].

Definition 2.15 (Multi-cycle holosymmetric valuation)**.** If the dual-history comparison is repeated for T successive QEC cycles (with appropriate coherent wiring of the reference system), we denote the resulting holosymmetric valuation by

$$v_{\mathrm{Holo}}^{(L,T)} \in [0, 1]. \tag{7}$$

In many models of interest one can write $v_{\mathrm{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + C^{(L,T)}\Big)$ for a suitable process-overlap scalar $C^{(L,T)}$, with T scaling polynomially in the relevant code parameters in threshold settings [2, 3, 4].

Definition 2.16 (Process-equivalence proposition)**.** For each code size L, with two QEC schedules $\mathcal{N}_A^{(L)}$ and $\mathcal{N}_B^{(L)}$ as in Definition 2.11, we define the *process-equivalence proposition* to be the statement

$$\varphi_{\mathrm{proc}}^{(L)} : \quad \text{“The A and B histories on } \mathcal{C}_L \text{ are holosymmetrically equivalent.”} \tag{8}$$

We regard the holosymmetric valuation $v_{\mathrm{Holo}}^{(L)}$ as the Thesis valuation in the “process corner” of the Tetralemma, denoted

$$v_{\mathrm{Holo}}^{(L)} = v_{\mathrm{Th,proc}}\Big(\rho_{\mathrm{out}}^{(L)}; \varphi_{\mathrm{proc}}^{(L)}\Big),$$

in line with the Fourfold Dialectic semantics for process-level propositions introduced in [8, Chs. 1–2].

Remark 2.17. In a full Tetralemma semantics for process propositions, the AntiThesis, Synthesis, and Holothesis valuations would be derived from complementary effects, such as the outcome $R = 1$ in the interferometer or decohered dual histories, following the general scheme of dual-register Holosymmetry in [8, Ch. 2]. For the fault-tolerance threshold theorem, however, we will only need to constrain the Thesis component $v_{\mathrm{Holo}}^{(L,T)}$.

2.4 Target Threshold Claim in Valuation Language

We now state the shape of the threshold claim in terms of the logical and holosymmetric valuations introduced above. The detailed hypotheses (local noise model, correlator bounds, etc.) will be spelled out in subsequent sections; here we simply record the structure we aim to prove, in analogy with standard fault-tolerance threshold theorems [2, 17, 3, 1, 4, 5], and expressed in the Tetralemma valuation language of [8, Chs. 1–2].

Definition 2.18 (Physical noise parameters (informal))**.** For each code size L we will extract, from the behaviour of the Fourfold correlator $\hat{G}_\Lambda$ and its induced channels on the Twofold sector as developed in [8, Ch. 4], together with the emergent-register constructions of [8, Chs. 9–10], the following effective noise parameters, mirroring the usual local-noise parameterizations in threshold analyses [5, 1, 2]:

- $p \in [0, 1]$: a *physical error rate per PQS per QEC cycle* in the Twofold register, derived from the effective local channel on each $\mathcal{H}_{(0,1),x_j}$ obtained from the Twofold block of $\hat{G}_\Lambda$ [8, Ch. 4];
- $\delta \in [0, 1]$: a *differential noise parameter* between schedules A and B, governing the overlap of their single-PQS error distributions (e.g. via a Bhattacharyya coefficient), interpreted as a process-level contrast parameter in the holosymmetric sense of [8, Ch. 2];
- $\varepsilon_{\mathrm{ps}} \geq 0$: a *leakage parameter*, measuring how strongly the dynamics per cycle mix the Twofold register $\mathcal{H}_{\mathrm{reg}}^{(L)}$ with the complementary (p, s)-sectors of the modal fibres, i.e. leakage out of the Twofold sector into parasymmetric/holosymmetric modes in the sense of the Fourfold Dialectic [8, Ch. 4].

Theorem 2.19 (Target IQR fault-tolerance threshold theorem (informal))**.** *Assume the ambient IQR setting of Assumption 2.1, together with a family of surface-code patches $\{\mathcal{C}_L\}_L$ as in Definition 2.3, and for each L a pair of QEC schedules A, B as in Definition 2.11 [16, 21, 1, 4, 15, 8].*

Suppose the effective physical noise parameters $p, \delta, \varepsilon_{\mathrm{ps}}$ (as in Definition 2.18) satisfy

$$p < p_*, \qquad \delta < \delta_*, \qquad \varepsilon_{\mathrm{ps}} < \varepsilon_{\mathrm{ps},*} \tag{9}$$

for some positive constants $p_*, \delta_*, \varepsilon_{\mathrm{ps},*} > 0$, independent of L [2, 17, 3, 1, 22, 4].

Then, for any target accuracy $\epsilon > 0$, there exists a code size L (with code distance $d_L \sim L$) such that:

(a) Logical correctness is high. *The Thesis valuation of the logical correctness proposition* $\varphi_L^{(L)}$ *satisfies*

$$\upsilon_{\mathrm{Th}}(\rho_{\mathrm{out}};\varphi_L^{(L)}) > 1-\epsilon \quad\Longleftrightarrow\quad p_L^{(L)} < \epsilon, \tag{10}$$

with $p_L^{(L)}$ *decaying as a suitable function of the distance* d_L *once* $p < p_*$ *[2, 3, 1, 4], and* υ_{Th} *interpreted as the Thesis component of the Tetralemma valuation on state-level propositions [8, Chs. 1–2].*

(b) Process equivalence is high over many cycles. *For the process-equivalence proposition* $\varphi_{\mathrm{proc}}^{(L)}$*, the multi-cycle holosymmetric valuation satisfies*

$$\upsilon_{\mathrm{Holo}}^{(L,T)} > 1-\epsilon \qquad \textit{for all} \quad T \le \mathrm{poly}(d_L), \tag{11}$$

where $\mathrm{poly}(d_L)$ *denotes some fixed polynomial in the code distance, as is standard in threshold formulations of long-time fault-tolerant computation [2, 3, 4], and* $\upsilon_{\mathrm{Holo}}^{(L,T)}$ *is the holosymmetric Thesis valuation in the process corner of the Tetralemma, in the sense of[8, Ch. 2].*

Equivalently, in Tetralemma semantics [8, Chs. 1–2]:

- *For the state-level proposition* $\varphi_L^{(L)}$*, the Thesis valuation* υ_{Th} *exceeds* $1-\epsilon$*, and the combined weight of the AntiThesis, Synthesis, and Holothesis valuations is less than* ϵ*.*
- *For the process-level proposition* $\varphi_{\mathrm{proc}}^{(L)}$*, the holosymmetric Thesis valuation* $\upsilon_{\mathrm{Holo}}^{(L,T)}$ *exceeds* $1-\epsilon$ *for all* $T \le \mathrm{poly}(d_L)$*, and the total weight assigned to alternative corners ("A and B behave distinctly", or "the A/B distinction is ill-posed") is less than* ϵ*.*

Remark 2.20. The detailed proof of Theorem 2.19 will rely on:

- extracting effective local Pauli noise models on the Twofold register from the Fourfold correlator and the Holo-State, in the spirit of the emergent-register and noise constructions in [8, Chs. 4,9–10] and of stabilizer and Pauli-noise analyses of fault tolerance [13, 5];
- applying surface-code threshold estimates for the logical error $p_L^{(L)}$ as a function of the physical error rate p and the code distance d_L [1, 22, 4, 12];
- using the holosymmetry process-overlap formulas, based on the holosymmetric control and valuation machinery of [8, Ch. 2], to control $\upsilon_{\mathrm{Holo}}^{(L,T)}$ in terms of a differential noise parameter δ and the number of PQS touched per cycle, in analogy with process-level robustness considerations in topological and surface-code architectures [16, 11, 15].

Phase 0 of the construction consists precisely in fixing the statements and valuation maps that the theorem refers to; subsequent phases will provide the necessary technical machinery and bounds, drawing on the IQR chronogeometric and dynamical structures of [8, Chs. 9–10,15].

3 Phase 1: Emergent Qubit Register and Leakage Control

In this section we make precise the construction of the emergent Twofold (qubit) register and introduce a quantitative notion of leakage from this register into the complementary (p, s)-sectors during a noisy QEC cycle, in the spirit of standard quantum error-correction and fault-tolerance frameworks [18, 19, 20, 13, 5], and using the Fourfold/Tetralemma formalism of [8, Chs. 1–2,4].

We work throughout under the ambient assumptions of Section 2, in particular Assumption 2.1 [8, Chs. 1–2,4].

3.1 Twofold Register and (p, s) Complement as Sub-Bundles

We first formalize the realization of ontological modes as rays in the modal fibres and the induced decomposition of each fibre into a Twofold subspace and its (p, s)-complement, in line with the Fourfold Dialectic and modal-bundle constructions of [8, Ch. 1].

Definition 3.1 (Ontological mode set)**.** Let

$$\mathfrak{M} := \left\{ \mathring{\mathbf{0}}, \mathring{\mathbf{1}}, \mathring{\mathbf{p}}, \mathring{\mathbf{s}} \right\}$$

denote the Fourfold set of ontological modes (Thesis, Antithesis, Parasynthesis, Holosynthesis) as introduced in the Tetralemma semantics of [8, Chs. 1–2].

Definition 3.2 (Realization map at a PQS locus)**.** For each PQS $\mathfrak{B}_x$, let $\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)} \cong \mathbb{C}^4$ be its modal fibre (Assumption 2.1). A *realization map* at x is a map

$$\mathcal{R}_x : \mathfrak{M} \to \mathbb{P}\Big(\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)}\Big), \tag{12}$$

from the ontological mode set to the projective space of the fibre, assigning to each ontological label $\mathring{\mathbf{m}} \in \mathfrak{M}$ a ray $[\psi_{\mathbf{m},x}] \subset \mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)}$.

For each mode we choose an arbitrary but fixed non-zero representative vector $|\psi_{\mathbf{m},x}\rangle \in \mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)}$ of the corresponding ray. This is the local realization of the Fourfold modes as modal rays, as in the Tetralemma–GNS construction of [8, Ch. 1].

Assumption 3.3 (Coincident-limit inner product)**.** *For each $x \in \mathcal{X}_q$, the Fourfold correlator $\hat{G}_\Lambda(\mathfrak{B}_x, \mathfrak{B}_{x'})$ and the Holo-State $\omega^{\mathring{\mathbf{\Psi}}}$ admit a well-defined* coincident limit*, in the sense that there exists a positive semidefinite sesquilinear form*

$$\langle \cdot, \cdot \rangle_x : \mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)} \times \mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)} \to \mathbb{C}$$

such that, for all ϕ_1, ϕ_2 in a dense subspace of $\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)}$,

$$\langle \phi_1, \phi_2 \rangle_x = \lim_{x' \to x} \Big\langle \Omega_\Psi \Big| \hat{G}_\Lambda(\mathfrak{B}_x, \mathfrak{B}_{x'})[\phi_1^*, \phi_2] \Big| \Omega_\Psi \Big\rangle, \tag{13}$$

and this form extends uniquely to a positive-definite inner product on $\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)}$ [8, Ch. 4].

Remark 3.4. Assumption 3.3 encodes the idea that the fibre Hilbert structure is *induced* from Holo-State expectations of the Fourfold correlator in the coincident limit $x' \to x$, as developed in the Fourfold/Tetralemma treatment of modal fibres in [8, Ch. 4].

Definition 3.5 (Fourfold realized frame and Gram matrix)**.** Fix $x \in \mathcal{X}_q$, and let

$$|\psi_{0,x}\rangle := |\psi^{\circ}_{\mathbf{0},x}\rangle, \quad |\psi_{1,x}\rangle := |\psi^{\circ}_{\mathbf{1},x}\rangle, \quad |\psi_{p,x}\rangle := |\psi^{\circ}_{\mathbf{p},x}\rangle, \quad |\psi_{s,x}\rangle := |\psi^{\circ}_{\mathbf{s},x}\rangle$$

be representatives of the realized rays for the four ontological modes.

The *Fourfold realized frame* at x is the ordered quadruple

$$\mathcal{B}_x^{\mathrm{raw}} := \big(|\psi_{0,x}\rangle, |\psi_{1,x}\rangle, |\psi_{p,x}\rangle, |\psi_{s,x}\rangle\big).$$

The corresponding *Gram matrix* is the 4×4 matrix $M(x) = [M_{\alpha\beta}(x)]_{\alpha,\beta\in\{0,1,p,s\}}$ with entries

$$M_{\alpha\beta}(x) := \langle \psi_{\alpha,x}, \psi_{\beta,x}\rangle_x. \tag{14}$$

This matches the Gram construction used to orthonormalize the Fourfold frame in [8, Ch. 4].

Assumption 3.6 (Non-degeneracy of the Gram matrix)**.** *For each x in the region of interest $\mathcal{U} \subset \mathcal{X}_q$, the Gram matrix $M(x)$ is positive-definite, hence invertible. Moreover, the map $x \mapsto M(x)$ is smooth (or at least continuous) on $\mathcal{U}$, in the sense required for the smooth modal-bundle structure in [8, Ch. 4].*

Lemma 3.7 (Orthonormal Fourfold basis)**.** *Under Assumptions 3.3 and 3.6, for each $x \in \mathcal{U}$ there exist vectors $|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x \in \mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)}$ such that:*

(a) The set $\mathcal{B}_x := \{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$ is an orthonormal basis of $\mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)}$ with respect to $\langle\cdot,\cdot\rangle_x$:

$$\langle \alpha, \beta\rangle_x = \delta_{\alpha\beta}, \qquad \alpha, \beta \in \{0, 1, p, s\}. \tag{15}$$

(b) Each basis vector $|\alpha\rangle_x$ lies in the span of the realized frame $\mathcal{B}_x^{\mathrm{raw}}$:

$$|\alpha\rangle_x = \sum_{\beta\in\{0,1,p,s\}} [U(x)]_{\beta\alpha}\, |\psi_{\beta,x}\rangle \tag{16}$$

for some $U(x) \in \mathrm{U}(4)$ depending smoothly on $x \in \mathcal{U}$.

Proof. Fix $x \in \mathcal{U}$. For brevity write $\mathcal{H}_x := \mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)}$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_x$.

Step 1: Linear independence and spanning of the raw frame. By construction (Definition 3.5), the Gram matrix $M(x)$ is defined by

$$M_{\alpha\beta}(x) := \langle \psi_{\alpha,x}, \psi_{\beta,x}\rangle, \qquad \alpha, \beta \in \{0, 1, p, s\}.$$

Assumption 3.6 says that $M(x)$ is positive-definite. In particular, for any non-zero vector $a = (a_\alpha)_{\alpha\in\{0,1,p,s\}} \in \mathbb{C}^4$,

$$a^\dagger M(x) a = \sum_{\alpha,\beta} \overline{a_\alpha} M_{\alpha\beta}(x) a_\beta = \left\langle \sum_\alpha a_\alpha \psi_{\alpha,x}\ ,\ \sum_\beta a_\beta \psi_{\beta,x} \right\rangle > 0.$$

If the realized frame $\{\psi_{\alpha,x}\}$ were linearly dependent, there would exist a non-zero a with $\sum_\alpha a_\alpha \psi_{\alpha,x} = 0$, which would give $a^\dagger M(x) a = 0$, contradicting positive-definiteness. Hence $\{\psi_{\alpha,x}\}_{\alpha\in\{0,1,p,s\}}$ is linearly independent.

Since $\dim \mathcal{H}_x = 4$ by Assumption 2.1 (see the modal-fibre construction in [8, Ch. 4]), the four linearly independent vectors $\psi_{\alpha,x}$ form a basis of $\mathcal{H}_x$. In particular, the linear map

$$T(x) : \mathbb{C}^4 \to \mathcal{H}_x, \qquad T(x)\,|e_\alpha\rangle := |\psi_{\alpha,x}\rangle,$$

is an isomorphism of complex vector spaces, where $\{|e_\alpha\rangle\}$ is the standard coordinate basis of $\mathbb{C}^4$.

Step 2: Encoding the inner product via the Gram matrix. For arbitrary $u, v \in \mathbb{C}^4$, write $u = \sum_\alpha u_\alpha |e_\alpha\rangle$, $v = \sum_\beta v_\beta |e_\beta\rangle$. Then

$$T(x)u = \sum_\alpha u_\alpha \psi_{\alpha,x}, \qquad T(x)v = \sum_\beta v_\beta \psi_{\beta,x}.$$

Using the definition of $M(x)$,

$$\begin{aligned}
\langle T(x)u, T(x)v\rangle &= \left\langle \sum_\alpha u_\alpha \psi_{\alpha,x}, \sum_\beta v_\beta \psi_{\beta,x} \right\rangle \\
&= \sum_{\alpha,\beta} \overline{u_\alpha} v_\beta \langle \psi_{\alpha,x}, \psi_{\beta,x}\rangle \\
&= \sum_{\alpha,\beta} \overline{u_\alpha} M_{\alpha\beta}(x) v_\beta \\
&= \langle u, M(x)v\rangle_{\mathbb{C}^4},
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^4}$ is the standard Hermitian inner product on $\mathbb{C}^4$.

Thus $T(x)$ is an isomorphism of vector spaces whose pull-back of the fibre inner product is exactly the Gram form $M(x)$ on $\mathbb{C}^4$.

Step 3: Constructing an isometry $U(x)$ via the positive square root. By Assumption 3.6, $M(x)$ is positive-definite. Therefore there exists a unique positive-definite self-adjoint square root $M(x)^{1/2}$ and its inverse $M(x)^{-1/2}$, both acting on $\mathbb{C}^4$. (Existence and uniqueness of $M^{1/2}$ for positive-definite matrices is a standard finite-dimensional fact, e.g. by spectral decomposition.)

Define

$$U(x) := T(x)\,M(x)^{-1/2} : \mathbb{C}^4 \to \mathcal{H}_x.$$

For $u, v \in \mathbb{C}^4$,

$$\begin{aligned}
\langle U(x)u, U(x)v\rangle &= \langle T(x)M(x)^{-1/2}u,\ T(x)M(x)^{-1/2}v\rangle \\
&= \left\langle M(x)^{-1/2}u,\ M(x)\,M(x)^{-1/2}v \right\rangle_{\mathbb{C}^4} \quad \text{(by the previous step)} \\
&= \langle M(x)^{-1/2}u,\ M(x)^{1/2}v\rangle_{\mathbb{C}^4} \\
&= \langle u, v\rangle_{\mathbb{C}^4},
\end{aligned}$$

since $M(x)^{-1/2}$ and $M(x)^{1/2}$ are adjoint inverses on $\mathbb{C}^4$.

Hence $U(x)$ is an isometry from $(\mathbb{C}^4, \langle\cdot,\cdot\rangle_{\mathbb{C}^4})$ into $(\mathcal{H}_x, \langle\cdot,\cdot\rangle)$. Because $T(x)$ is an isomorphism and $M(x)^{-1/2}$ is invertible, $U(x)$ is bijective as a linear map $\mathbb{C}^4 \to \mathcal{H}_x$. An isometric bijection between finite-dimensional Hilbert spaces is unitary. Thus $U(x) \in \mathrm{U}(4)$.

Step 4: Defining the orthonormal Fourfold basis. Let $\{|e_\alpha\rangle\}_{\alpha\in\{0,1,p,s\}}$ be the standard orthonormal basis of $(\mathbb{C}^4, \langle\cdot,\cdot\rangle_{\mathbb{C}^4})$. Define

$$|\alpha\rangle_x := U(x)|e_\alpha\rangle, \qquad \alpha \in \{0, 1, p, s\}.$$

Since $U(x)$ is unitary, the set $\mathcal{B}_x := \{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$ is an orthonormal basis of $\mathcal{H}_x$:

$$\langle\alpha, \beta\rangle_x = \langle U(x)e_\alpha, U(x)e_\beta\rangle = \langle e_\alpha, e_\beta\rangle_{\mathbb{C}^4} = \delta_{\alpha\beta}.$$

This proves part (a).

Moreover, each $|\alpha\rangle_x$ lies in the span of the realized frame $\{\psi_{\beta,x}\}$, because $U(x)$ is built from $T(x)$, whose image is $\operatorname{span}\{\psi_{\beta,x}\}$:

$$|\alpha\rangle_x = U(x)|e_\alpha\rangle = T(x)M(x)^{-1/2}|e_\alpha\rangle = \sum_{\beta\in\{0,1,p,s\}} [U(x)]_{\beta\alpha}\, |\psi_{\beta,x}\rangle,$$

where we have expanded $M(x)^{-1/2}|e_\alpha\rangle$ in the coordinate basis, and $[U(x)]_{\beta\alpha}$ are the matrix elements of $U(x)$ in the realized frame. This proves part (b) at the level of linear combinations.

Step 5: Smooth dependence on x. By Assumption 3.3, the inner product $\langle\cdot,\cdot\rangle_x$ is induced from Holo-State expectations of the Fourfold correlator in the coincident limit (13), and the construction of the modal fibres in [8, Ch. 4] ensures that the realized mode vectors $|\psi_{\alpha,x}\rangle$ and the inner products $\langle\psi_{\alpha,x}, \psi_{\beta,x}\rangle_x$ depend smoothly on $x \in \mathcal{U}$ (at least on the region where the Gram matrix remains non-degenerate).

Thus each entry $M_{\alpha\beta}(x)$ is a smooth function of x, and the map $x \mapsto M(x)$ is smooth on $\mathcal{U}$. The map

$$\mathrm{PosDef}(4) \to \mathrm{PosDef}(4), \qquad A \mapsto A^{-1/2},$$

from the cone of 4 × 4 positive-definite matrices to itself, is smooth (indeed real-analytic), e.g. by the standard holomorphic functional calculus on the spectrum of A. Composing $x \mapsto M(x)$ with this smooth map yields a smooth $x \mapsto M(x)^{-1/2}$.

Similarly, the map $x \mapsto T(x)$ is smooth, because the realized frame vectors $|\psi_{\alpha,x}\rangle = T(x)|e_\alpha\rangle$ vary smoothly with x by the modal-bundle construction in [8, Ch. 4]. Therefore the product

$$U(x) = T(x)M(x)^{-1/2}$$

depends smoothly on $x \in \mathcal{U}$ as a composition and product of smooth maps. This gives the asserted smooth dependence of the coefficients $[U(x)]_{\beta\alpha}$ and hence of the basis vectors $|\alpha\rangle_x$.

Combining Steps 1–5, we have constructed, for each $x \in \mathcal{U}$, an orthonormal basis $\mathcal{B}_x$ of the fibre $\mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)}$ with the desired properties, completing the proof. □

Definition 3.8 (Twofold and (p, s) subspaces)**.** For each $x \in \mathcal{U}$, define the Twofold (symmetry–antisymmetry) subspace

$$\mathcal{H}_{(0,1),x} := \operatorname{span}\Big\{|0\rangle_x, |1\rangle_x\Big\} \subset \mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)},$$

and the (p, s) subspace

$$\mathcal{H}_{(p,s),x} := \operatorname{span}\Big\{|p\rangle_x, |s\rangle_x\Big\} \subset \mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)}.$$

We then have an orthogonal direct sum decomposition

$$\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)} = \mathcal{H}_{(0,1),x} \oplus \mathcal{H}_{(p,s),x}, \tag{17}$$

in accordance with the Twofold versus parasymmetric/holosymmetric splitting of the modal fibres in [8, Ch. 4].

Lemma 3.9 (Smooth sub-bundle structure)**.** *Under the hypotheses of Lemma 3.7, the assignments*

$$x \mapsto \mathcal{H}_{(0,1),x}, \qquad x \mapsto \mathcal{H}_{(p,s),x},$$

define rank-2 complex sub-bundles of the modal Hilbert bundle $\mathcal{H}^{\text{modal}} \to \mathcal{U}$*, and the decomposition*

$$\mathcal{H}^{\text{modal}}|_{\mathcal{U}} \cong \mathcal{H}_{(0,1)} \oplus \mathcal{H}_{(p,s)}$$

is smooth over $\mathcal{U}$*.*

Proof. We recall that $\mathcal{H}^{\text{modal}} \to \mathcal{U}$ is, by construction, a rank-4 complex vector bundle whose fibre at x is $\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)} \cong \mathbb{C}^4$, with the fibre-wise inner product $\langle \cdot, \cdot \rangle_x$ induced from the Fourfold correlator in the coincident limit [8, Ch. 4].

Step 1: A smooth orthonormal frame gives a smooth trivialization.

By Lemma 3.7, for each $x \in \mathcal{U}$ there is an orthonormal basis

$$\mathcal{B}_x = \{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$$

of the fibre $\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)}$, such that:

- $\langle \alpha, \beta \rangle_x = \delta_{\alpha\beta}$ for $\alpha, \beta \in \{0, 1, p, s\}$; and
- each $|\alpha\rangle_x$ depends smoothly on $x \in \mathcal{U}$, because the unitary $U(x) \in \mathrm{U}(4)$ constructed in the proof of Lemma 3.7 depends smoothly on x.

Fix once and for all the standard orthonormal basis $\{|e_0\rangle, |e_1\rangle, |e_p\rangle, |e_s\rangle\}$ of $\mathbb{C}^4$. Define, for each $x \in \mathcal{U}$, the linear isomorphism

$$T(x) : \mathbb{C}^4 \to \mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)}, \qquad T(x)|e_\alpha\rangle := |\alpha\rangle_x.$$

Because the basis $\mathcal{B}_x$ depends smoothly on x, the map $x \mapsto T(x)$ is smooth in the sense of vector bundles: in local coordinates on $\mathcal{U}$, each matrix element $\langle \psi, T(x)\phi \rangle_x$ is a smooth function of x for fixed ψ, ϕ.

Now define a map

$$\Phi : \mathcal{U} \times \mathbb{C}^4 \longrightarrow \mathcal{H}^{\text{modal}}|_{\mathcal{U}}, \qquad \Phi(x, z) := (x, T(x)z).$$

This map is:

- a bijection fibrewise (for each fixed x, $T(x)$ is an isomorphism $\mathbb{C}^4 \to \mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)}$);
- smooth in both x and z, since $T(x)$ depends smoothly on x and linearly on z;
- a vector-bundle isomorphism, since each $\Phi_x := T(x)$ is linear.

Thus Φ gives a smooth trivialization

$$\mathcal{H}^{\text{modal}}|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{C}^4,$$

with the identification

$$(x, z_0, z_1, z_p, z_s) \longleftrightarrow z_0|0\rangle_x + z_1|1\rangle_x + z_p|p\rangle_x + z_s|s\rangle_x.$$

Step 2: Defining the sub-bundles in the trivialization.

Within the trivial bundle $\mathcal{U} \times \mathbb{C}^4$, consider the two subsets

$$\begin{aligned} E_{(0,1)} &:= \Big\{(x, z) \in \mathcal{U} \times \mathbb{C}^4 : z = (z_0, z_1, 0, 0) \in \mathbb{C}^4\Big\} \\ &= \mathcal{U} \times \Big(\text{span}\{|e_0\rangle, |e_1\rangle\}\Big), \end{aligned} \tag{18}$$

$$\begin{aligned} E_{(p,s)} &:= \Big\{(x, z) \in \mathcal{U} \times \mathbb{C}^4 : z = (0, 0, z_p, z_s) \in \mathbb{C}^4\Big\} \\ &= \mathcal{U} \times \Big(\text{span}\{|e_p\rangle, |e_s\rangle\}\Big). \end{aligned} \tag{19}$$

Each of these is evidently a rank-2 complex vector sub-bundle of $\mathcal{U} \times \mathbb{C}^4$, since the fibres span$\{|e_0\rangle, |e_1\rangle\}$ and span$\{|e_p\rangle, |e_s\rangle\}$ do not depend on x and give constant-rank subspaces of $\mathbb{C}^4$.

Under the trivialization Φ, we transport these sub-bundles back to $\mathcal{H}^{\text{modal}}|_{\mathcal{U}}$ by defining

$$\mathcal{H}_{(0,1)} := \Phi\Big(E_{(0,1)}\Big) = \Big\{(x, z_0|0\rangle_x + z_1|1\rangle_x) : x \in \mathcal{U},\ z_0, z_1 \in \mathbb{C}\Big\}, \tag{20}$$

$$\mathcal{H}_{(p,s)} := \Phi\Big(E_{(p,s)}\Big) = \Big\{(x, z_p|p\rangle_x + z_s|s\rangle_x) : x \in \mathcal{U},\ z_p, z_s \in \mathbb{C}\Big\}. \tag{21}$$

By construction, the fibre of $\mathcal{H}_{(0,1)}$ at x is

$$\mathcal{H}_{(0,1),x} = \text{span}\Big\{|0\rangle_x, |1\rangle_x\Big\},$$

and similarly the fibre of $\mathcal{H}_{(p,s)}$ at x is

$$\mathcal{H}_{(p,s),x} = \text{span}\Big\{|p\rangle_x, |s\rangle_x\Big\},$$

as in Definition 3.8. Since $E_{(0,1)}$ and $E_{(p,s)}$ are smooth sub-bundles of the trivial bundle and Φ is a smooth bundle isomorphism, it follows that $\mathcal{H}_{(0,1)}$ and $\mathcal{H}_{(p,s)}$ are rank-2 complex sub-bundles of $\mathcal{H}^{\text{modal}}|_{\mathcal{U}}$.

Step 3: Smoothness of the direct-sum decomposition.

Pointwise, for each $x \in \mathcal{U}$, the orthonormal basis $\{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$ guarantees the orthogonal direct sum

$$\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_x)} = \mathcal{H}_{(0,1),x} \oplus \mathcal{H}_{(p,s),x}.$$

Indeed, any vector $v \in \mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}}$ has a unique decomposition

$$v = \Big(z_0|0\rangle_x + z_1|1\rangle_x\Big) + \Big(z_p|p\rangle_x + z_s|s\rangle_x\Big),$$

with complex coefficients (z_0, z_1, z_p, z_s), and the two summands are orthogonal because the basis is orthonormal.

At the bundle level, under the trivialization $\mathcal{H}^{\mathrm{modal}}|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{C}^4$, this decomposition is simply

$$\mathcal{U} \times \mathbb{C}^4 = E_{(0,1)} \oplus E_{(p,s)},$$

a direct sum of trivial rank-2 bundles. Transporting back via Φ, we obtain the smooth direct-sum decomposition

$$\mathcal{H}^{\mathrm{modal}}|_{\mathcal{U}} \cong \mathcal{H}_{(0,1)} \oplus \mathcal{H}_{(p,s)}.$$

This shows that:

- $x \mapsto \mathcal{H}_{(0,1),x}$ and $x \mapsto \mathcal{H}_{(p,s),x}$ define rank-2 complex sub-bundles of the modal Hilbert bundle; and
- the decomposition into Twofold and (p, s) sectors is a smooth decomposition of vector bundles over $\mathcal{U}$.

Thus the lemma is proved. □

Definition 3.10 (Emergent Twofold register for a finite patch)**.** Let $\{x_j\}_{j=1}^n \subset \mathcal{U}$ be a finite configuration of PQS loci, interpreted as a patch of the PQS continuum. The *emergent Twofold register* on this patch is the tensor-product Hilbert space

$$\mathcal{H}_{\mathrm{reg}} := \bigotimes_{j=1}^{n} \mathcal{H}_{(0,1),x_j} \;\cong\; (\mathbb{C}^2)^{\otimes n}, \tag{22}$$

realizing an n-qubit register embedded in the IQR modal bundle, in direct analogy with standard qubit-register constructions in quantum error-correcting codes [18, 19, 20, 13, 5] and with the emergent-register constructions in [8, Chs. 4,9–10].

3.2 Leakage Parameter from Twofold to (p, s) Sectors

We now define a notion of leakage from the emergent Twofold register into the complementary (p, s) sectors during one QEC cycle, in line with standard treatments of leakage and subsystem dynamics in fault-tolerant quantum computation [18, 20, 13, 3, 5], and using the Twofold/(p, s) modal sector decomposition and emergent-register embeddings developed in [8, Chs. 4,9–10].

Definition 3.11 (Local projectors onto Twofold and (p, s) sectors)**.** For each $x \in \mathcal{U}$, let $P_{(0,1)}(x)$ and $P_{(p,s)}(x)$ denote the orthogonal projections on $\mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}}$ onto $\mathcal{H}_{(0,1),x}$ and $\mathcal{H}_{(p,s),x}$, respectively:

$$P_{(0,1)}(x) := |0\rangle_x\langle 0|_x + |1\rangle_x\langle 1|_x, \qquad P_{(p,s)}(x) := |p\rangle_x\langle p|_x + |s\rangle_x\langle s|_x. \tag{23}$$

These are the fibrewise sector projectors associated with the Twofold and (p, s) subspaces identified in the Fourfold modal decomposition of Chapter 4 of [8].

Definition 3.12 (Global register projector)**.** Let $\{x_j\}_{j=1}^n \subset \mathcal{U}$ define a patch and $\mathcal{H}_{\text{reg}}$ the corresponding emergent register (Definition 3.10). We regard the total Hilbert space of all PQS in the patch (and possibly additional degrees of freedom) as an orthogonal direct sum

$$\mathcal{H}_{\text{tot}} \cong \mathcal{H}_{\text{reg}} \oplus \mathcal{H}_{\text{comp}},$$

where $\mathcal{H}_{\text{comp}}$ collects all modes outside the tensor product of Twofold subspaces at the sites x_j.

Define the orthogonal projection

$$P_{\text{reg}} : \mathcal{H}_{\text{tot}} \to \mathcal{H}_{\text{reg}} \tag{24}$$

onto $\mathcal{H}_{\text{reg}}$, and its orthogonal complement

$$P_{\text{comp}} := \mathbb{I}_{\text{tot}} - P_{\text{reg}}. \tag{25}$$

This matches the way the Twofold register is embedded as a distinguished subspace of the larger emergent Hilbert space in the register constructions of [8, Chs. 9–10].

Remark 3.13. Concretely, if we ignore all degrees of freedom outside the patch $\{x_j\}_{j=1}^n$, one may take

$$P_{\text{reg}} = \bigotimes_{j=1}^{n} P_{(0,1)}(x_j), \qquad P_{\text{comp}} = \mathbb{I} - P_{\text{reg}}$$

on the tensor product $\bigotimes_{j=1}^n \mathcal{H}_{\text{mod}}^{(\mathfrak{B}_{x_j})}$. In a more complete treatment $\mathcal{H}_{\text{tot}}$ includes additional ancilla and environment factors; the abstract P_{reg} projects onto the subspace isomorphic to $\mathcal{H}_{\text{reg}}$, as in standard QEC constructions where the code subspace is embedded in a larger physical Hilbert space [13, 5]. In the IQR setting this is precisely the embedding of the Twofold register inside the emergent arena described in [8, Chs. 9–10].

Definition 3.14 (Restriction of the dynamics to the register)**.** Let $\mathcal{N} : \mathcal{S}(\mathcal{H}_{\text{tot}}) \to \mathcal{S}(\mathcal{H}_{\text{tot}})$ be a CPTP map describing one QEC cycle on the total system, in the usual Kraus-channel sense [5], and arising from the IQR/Holo-State dynamics on $\mathcal{H}_{\text{em}}$ restricted to the relevant degrees of freedom as in [8, Chs. 9–10].

We regard states on $\mathcal{H}_{\text{reg}}$ as embedded into $\mathcal{H}_{\text{tot}}$ via the inclusion $\mathcal{H}_{\text{reg}} \hookrightarrow \mathcal{H}_{\text{tot}}$ and identify $\mathcal{S}(\mathcal{H}_{\text{reg}})$ with the subset of $\mathcal{S}(\mathcal{H}_{\text{tot}})$ supported in $\mathcal{H}_{\text{reg}}$.

The *reduced output state on the register* for an input $\rho \in \mathcal{S}(\mathcal{H}_{\text{reg}})$ is

$$\rho'_{\text{reg}} := \mathrm{Tr}_{\text{comp}}\Big[\mathcal{N}(\rho)\Big] \;\in\; \mathcal{S}(\mathcal{H}_{\text{reg}}), \tag{26}$$

where $\mathrm{Tr}_{\text{comp}}$ denotes the partial trace over $\mathcal{H}_{\text{comp}}$.

We also define the *confined register channel* $\mathcal{N}_{\text{reg}} : \mathcal{S}(\mathcal{H}_{\text{reg}}) \to \mathcal{S}(\mathcal{H}_{\text{reg}})$ by

$$\mathcal{N}_{\text{reg}}(\rho) := \mathrm{Tr}_{\text{comp}}\Big[P_{\text{reg}}\,\mathcal{N}(\rho)\,P_{\text{reg}}\Big]. \tag{27}$$

This is the standard subsystem-restricted channel obtained by compressing to the register subspace and tracing out the complement, exactly in the sense of the emergent subsystem reductions used in [8, Chs. 9–10,15].

Definition 3.15 (Leakage parameter)**.** The *leakage parameter* from the Twofold register into the complementary (p, s) sectors for one application of $\mathcal{N}$ is defined by

$$\varepsilon_{\text{ps}} := \sup_{\rho \in \mathcal{S}(\mathcal{H}_{\text{reg}})} \left\| \mathrm{Tr}_{\text{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}_{\text{reg}}(\rho) \right\|_1. \tag{28}$$

Remark 3.16. Intuitively, ε_{ps} measures how much the true reduced evolution of states supported on the Twofold register deviates, in trace norm, from an "idealized" evolution that remains strictly confined to the register subspace. In particular, if $\varepsilon_{\text{ps}} = 0$, then $\mathrm{Tr}_{\text{comp}}(\mathcal{N}(\rho)) = \mathcal{N}_{\text{reg}}(\rho)$ for all ρ and the dynamics does not generate any effective leakage beyond what is recaptured by the projection P_{reg}, closely mirroring leakage-free fault-tolerant noise models [3, 5]. In the IQR language, this corresponds to a situation in which the Twofold sector is dynamically closed under the emergent evolution generated from the Fourfold correlator and Holo-State, with no net flow into the (p, s) modes, as discussed in the sector dynamics of [8, Ch. 4].

We now relate ε_{ps} to off-block couplings in the microscopic dynamics, and establish a simple stability property under composition.

Assumption 3.17 (Block structure of the Kraus operators)**.** *Let $\mathcal{N}$ have a Kraus representation*

$$\mathcal{N}(\sigma) = \sum_i K_i \sigma K_i^\dagger, \qquad \sum_i K_i^\dagger K_i = \mathbb{I}_{\text{tot}},$$

with each K_i written in block form relative to $\mathcal{H}_{\text{tot}} \cong \mathcal{H}_{\text{reg}} \oplus \mathcal{H}_{\text{comp}}$:

$$K_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \qquad i \in I, \tag{29}$$

where $A_i : \mathcal{H}_{\text{reg}} \to \mathcal{H}_{\text{reg}}$, $B_i : \mathcal{H}_{\text{comp}} \to \mathcal{H}_{\text{reg}}$, $C_i : \mathcal{H}_{\text{reg}} \to \mathcal{H}_{\text{comp}}$, $D_i : \mathcal{H}_{\text{comp}} \to \mathcal{H}_{\text{comp}}$. Define

$$\eta_{\text{ps}} := \sup_{i \in I} \max\Big\{ \|B_i\|, \|C_i\| \Big\}, \tag{30}$$

where $\|\cdot\|$ denotes the operator norm.

Lemma 3.18 (Leakage bound in terms of off-block Kraus norms)**.** *Under Assumption 3.17, there exists a constant $C_{\text{dim}} > 0$, depending only on the dimension of $\mathcal{H}_{\text{reg}}$ and on the fixed local gadget architecture of one QEC cycle, such that*

$$\varepsilon_{\text{ps}} \;\le\; C_{\text{dim}}\, \eta_{\text{ps}}. \tag{31}$$

In particular, if η_{ps} is small, then the leakage parameter ε_{ps} is small.

Proof. Recall the leakage parameter from Definition 3.15:

$$\varepsilon_{\text{ps}} = \sup_{\rho \in \mathcal{S}(\mathcal{H}_{\text{reg}})} \left\| \mathrm{Tr}_{\text{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}_{\text{reg}}(\rho) \right\|_1,$$

where

$$\mathcal{N}_{\text{reg}}(\rho) = \mathrm{Tr}_{\text{comp}}\Big[P_{\text{reg}} \mathcal{N}(\rho) P_{\text{reg}}\Big].$$

Step 1: Expressing the difference via a CPTP map.

For $\rho \in \mathcal{S}(\mathcal{H}_{\text{reg}})$, viewed as a state on $\mathcal{H}_{\text{tot}}$ supported entirely in $\mathcal{H}_{\text{reg}}$, we can write

$$\rho = \begin{pmatrix} \rho_{\text{reg}} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_{\text{reg}} \in \mathcal{S}(\mathcal{H}_{\text{reg}}).$$

Define

$$\Delta(\rho) := \operatorname{Tr}_{\text{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}_{\text{reg}}(\rho) = \operatorname{Tr}_{\text{comp}}\Big(\mathcal{N}(\rho) - P_{\text{reg}}\mathcal{N}(\rho)P_{\text{reg}}\Big).$$

Since $\operatorname{Tr}_{\text{comp}}$ is a CPTP map (partial trace over $\mathcal{H}_{\text{comp}}$), it is trace-norm contractive. Thus

$$\|\Delta(\rho)\|_1 \;\le\; \Big\|\mathcal{N}(\rho) - P_{\text{reg}}\mathcal{N}(\rho)P_{\text{reg}}\Big\|_1. \tag{32}$$

Step 2: Block form of $\mathcal{N}(\rho)$ and the difference.

Under Assumption 3.17, we have a Kraus representation

$$\mathcal{N}(\sigma) = \sum_i K_i \sigma K_i^\dagger, \qquad \sum_i K_i^\dagger K_i = \mathbb{I}_{\text{tot}},$$

with each K_i written in block form relative to $\mathcal{H}_{\text{tot}} \cong \mathcal{H}_{\text{reg}} \oplus \mathcal{H}_{\text{comp}}$:

$$K_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}.$$

For our ρ supported in $\mathcal{H}_{\text{reg}}$,

$$K_i \rho K_i^\dagger = \begin{pmatrix} A_i \\ C_i \end{pmatrix} \rho_{\text{reg}} \begin{pmatrix} A_i^\dagger & C_i^\dagger \end{pmatrix} = \begin{pmatrix} A_i\rho_{\text{reg}}A_i^\dagger & A_i\rho_{\text{reg}}C_i^\dagger \\ C_i\rho_{\text{reg}}A_i^\dagger & C_i\rho_{\text{reg}}C_i^\dagger \end{pmatrix}.$$

Summing over i gives

$$\mathcal{N}(\rho) = \sum_i K_i \rho K_i^\dagger = \begin{pmatrix} \sum_i A_i\rho_{\text{reg}}A_i^\dagger & \sum_i A_i\rho_{\text{reg}}C_i^\dagger \\ \sum_i C_i\rho_{\text{reg}}A_i^\dagger & \sum_i C_i\rho_{\text{reg}}C_i^\dagger \end{pmatrix}.$$

Since P_{reg} is the projector onto $\mathcal{H}_{\text{reg}}$, in block form

$$P_{\text{reg}} = \begin{pmatrix} \mathbb{I}_{\text{reg}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{so} \quad P_{\text{reg}}\mathcal{N}(\rho)P_{\text{reg}} = \begin{pmatrix} \sum_i A_i\rho_{\text{reg}}A_i^\dagger & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$D(\rho) := \mathcal{N}(\rho) - P_{\text{reg}}\mathcal{N}(\rho)P_{\text{reg}} = \begin{pmatrix} 0 & X \\ Y & Z \end{pmatrix},$$

with

$$X := \sum_i A_i \rho_{\text{reg}} C_i^\dagger, \tag{33}$$

$$Y := \sum_i C_i \rho_{\text{reg}} A_i^\dagger, \tag{34}$$

$$Z := \sum_i C_i \rho_{\text{reg}} C_i^\dagger. \tag{35}$$

Combining with (32), we have

$$\|\Delta(\rho)\|_1 \ \leq \ \|D(\rho)\|_1. \tag{36}$$

Step 3: Bounding the trace norm of the block matrix.

We now bound $\|D(\rho)\|_1$ in terms of the norms of X, Y, Z. For any operator on a direct sum $\mathcal{H}_{\mathrm{reg}} \oplus \mathcal{H}_{\mathrm{comp}}$ with block form

$$\begin{pmatrix} 0 & X \\ Y & Z \end{pmatrix},$$

we have the crude but sufficient estimate

$$\left\| \begin{pmatrix} 0 & X \\ Y & Z \end{pmatrix} \right\|_1 \leq \|X\|_1 + \|Y\|_1 + \|Z\|_1. \tag{37}$$

Indeed, the left-hand side is the trace norm of the sum of three operators with disjoint block support:

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix},$$

so the triangle inequality yields (37).

Thus

$$\|D(\rho)\|_1 \leq \|X\|_1 + \|Y\|_1 + \|Z\|_1. \tag{38}$$

Step 4: Bounding the blocks X, Y, Z using η_{ps}.

By Assumption 3.17,

$$\eta_{\mathrm{ps}} = \sup_{i \in I} \max\{\|B_i\|, \|C_i\|\},$$

and from $\sum_i K_i^\dagger K_i = \mathbb{I}_{\mathrm{tot}}$ we have $\|K_i\| \leq 1$ for each i, hence in particular $\|A_i\| \leq 1$ and $\|C_i\| \leq 1$ for all i.

We use the standard inequality for trace-class operators:

$$\|X\rho Y^\dagger\|_1 \leq \|X\| \, \|Y\| \, \|\rho\|_1 \qquad \text{for bounded } X, Y \text{ and trace-class } \rho. \tag{39}$$

(For instance, this follows from Hölder inequalities and the duality of operator and trace norms.)

Using (39) and $\|\rho_{\mathrm{reg}}\|_1 = 1$:

(a) Off-diagonal blocks X and Y.

$$\begin{aligned} \|X\|_1 = \left\| \sum_i A_i \rho_{\mathrm{reg}} C_i^\dagger \right\|_1 &\leq \sum_i \|A_i \rho_{\mathrm{reg}} C_i^\dagger\|_1 \\ &\leq \sum_i \|A_i\| \, \|C_i\| \, \|\rho_{\mathrm{reg}}\|_1 \leq \sum_i \|C_i\| \\ &\leq (\#I) \, \eta_{\mathrm{ps}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \|Y\|_1 = \left\| \sum_i C_i \rho_{\mathrm{reg}} A_i^\dagger \right\|_1 &\leq \sum_i \|C_i \rho_{\mathrm{reg}} A_i^\dagger\|_1 \\ &\leq \sum_i \|C_i\| \, \|A_i\| \, \|\rho_{\mathrm{reg}}\|_1 \leq \sum_i \|C_i\| \\ &\leq (\#I) \, \eta_{\mathrm{ps}}. \end{aligned}$$

(b) Lower-right block Z.

$$\begin{aligned}\|Z\|_1 = \left\|\sum_i C_i \rho_{\mathrm{reg}} C_i^\dagger\right\|_1 &\le \sum_i \|C_i \rho_{\mathrm{reg}} C_i^\dagger\|_1 \\ &\le \sum_i \|C_i\|^2 \, \|\rho_{\mathrm{reg}}\|_1 \le (\#I)\,\eta_{\mathrm{ps}}^2.\end{aligned}$$

Combining with (38), we obtain

$$\|D(\rho)\|_1 \le 2(\#I)\,\eta_{\mathrm{ps}} + (\#I)\,\eta_{\mathrm{ps}}^2 = (\#I)\,\Big(2\eta_{\mathrm{ps}} + \eta_{\mathrm{ps}}^2\Big). \tag{40}$$

Step 5: Uniform linear bound in η_{ps}.

We now turn (40) into a bound linear in η_{ps}. First, note that for any CPTP map,

$$\|\Delta(\rho)\|_1 \le \|D(\rho)\|_1$$

by (36), and for any pair of quantum states the trace distance is at most 2. Hence for every ρ,

$$\|\Delta(\rho)\|_1 \le 2. \tag{41}$$

Consider two regimes:

(i) Small $\eta_{\mathrm{ps}} \le 1$. Then $\eta_{\mathrm{ps}}^2 \le \eta_{\mathrm{ps}}$, and

$$2\eta_{\mathrm{ps}} + \eta_{\mathrm{ps}}^2 \le 3\eta_{\mathrm{ps}}.$$

Thus (40) gives

$$\|\Delta(\rho)\|_1 \le \|D(\rho)\|_1 \le 3(\#I)\,\eta_{\mathrm{ps}}.$$

(ii) Large $\eta_{\mathrm{ps}} \ge 1$. Here we can use the trivial bound (41) together with $\eta_{\mathrm{ps}} \ge 1$:

$$\|\Delta(\rho)\|_1 \le 2 \le 2\,\eta_{\mathrm{ps}}.$$

Combining both regimes, we can write

$$\|\Delta(\rho)\|_1 \le C'\,\eta_{\mathrm{ps}},$$

for all $\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{reg}})$, where

$$C' := \max\Big\{3(\#I),\, 2\Big\}.$$

Step 6: Taking the supremum and encoding the local gadget architecture.

By definition of $\varepsilon_{\mathrm{ps}}$,

$$\varepsilon_{\mathrm{ps}} = \sup_{\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{reg}})} \|\Delta(\rho)\|_1 \le C'\,\eta_{\mathrm{ps}}.$$

To relate C' to the local gadget architecture, note that in finite dimension one can always choose a Kraus representation for $\mathcal{N}$ with at most $(\dim \mathcal{H}_{\mathrm{tot}})^2$ operators. In the fault-tolerant setting of interest, a single QEC cycle is implemented by a fixed-depth circuit of local gates and local noise channels, each acting on a constant-size neighbourhood of PQS loci and ancillas. Thus there exists a constant L_{gad} such that for any input supported in $\mathcal{H}_{\mathrm{reg}}$:

- at most L_{gad} Kraus operators have non-zero off-block components C_i (i.e. can contribute to leakage per cycle), and
- L_{gad} depends only on $\dim \mathcal{H}_{\mathrm{reg}}$ and the fixed local gadget architecture, not on the code size L.

Replacing $(\#I)$ by L_{gad} in the bounds above, we obtain a constant

$$C_{\mathrm{dim}} := \max\Big\{3L_{\mathrm{gad}},\, 2\Big\}$$

such that

$$\varepsilon_{\mathrm{ps}} \leq C_{\mathrm{dim}}\, \eta_{\mathrm{ps}},$$

with $C_{\mathrm{dim}} > 0$ depending only on the register dimension and the fixed local gadget architecture, and independent of the particular state ρ and the code size L.

This completes the proof. □

Lemma 3.19 (Stability of leakage under composition)**.** *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two CPTP maps on $\mathcal{H}_{\mathrm{tot}}$ with leakage parameters $\varepsilon_{\mathrm{ps}}^{(1)}$ and $\varepsilon_{\mathrm{ps}}^{(2)}$, respectively, defined as in Definition 3.15. For $i = 1, 2$, let $\mathcal{N}_{i,\mathrm{reg}}$ denote the associated confined register channel on $\mathcal{S}(\mathcal{H}_{\mathrm{reg}})$.*

Define the two-cycle composite map

$$\mathcal{N} := \mathcal{N}_2 \circ \mathcal{N}_1$$

on $\mathcal{S}(\mathcal{H}_{\mathrm{tot}})$, and take as its associated confined register channel the composition

$$\mathcal{N}_{\mathrm{reg}}^{(2\circ 1)} := \mathcal{N}_{2,\mathrm{reg}} \circ \mathcal{N}_{1,\mathrm{reg}} : \mathcal{S}(\mathcal{H}_{\mathrm{reg}}) \to \mathcal{S}(\mathcal{H}_{\mathrm{reg}}).$$

Then the two-cycle leakage parameter

$$\varepsilon_{\mathrm{ps}}^{(2\circ 1)} := \sup_{\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{reg}})} \Big\| \mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}_{\mathrm{reg}}^{(2\circ 1)}(\rho) \Big\|_1$$

satisfies

$$\varepsilon_{\mathrm{ps}}^{(2\circ 1)} \;\leq\; \varepsilon_{\mathrm{ps}}^{(1)} + \varepsilon_{\mathrm{ps}}^{(2)}. \tag{42}$$

In particular, for T identical cycles $\mathcal{N}^{(T)} := \mathcal{N}^T$, with associated confined register channel $\mathcal{N}_{\mathrm{reg}}^{(T)} := \mathcal{N}_{\mathrm{reg}}^T$, we have

$$\varepsilon_{\mathrm{ps}}^{(T)} := \sup_{\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{reg}})} \Big\| \mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}^{(T)}(\rho)\Big) - \mathcal{N}_{\mathrm{reg}}^{(T)}(\rho) \Big\|_1 \;\leq\; T\, \varepsilon_{\mathrm{ps}}, \tag{43}$$

where $\varepsilon_{\mathrm{ps}} := \varepsilon_{\mathrm{ps}}^{(1)}$ is the single-cycle leakage parameter. Thus leakage grows at most linearly with the number of cycles.

Proof. **Step 1: Single-cycle leakage in difference-map form.**

For $i = 1, 2$, Definition 3.15 states that there exists a CPTP map

$$\mathcal{N}_{i,\mathrm{reg}} : \mathcal{S}(\mathcal{H}_{\mathrm{reg}}) \to \mathcal{S}(\mathcal{H}_{\mathrm{reg}})$$

such that

$$\sup_{\rho\in\mathcal{S}(\mathcal{H}_{\mathrm{reg}})}\Big\|\mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}_i(\rho)\Big)-\mathcal{N}_{i,\mathrm{reg}}(\rho)\Big\|_1=\varepsilon^{(i)}_{\mathrm{ps}}. \tag{44}$$

For later use, define the *single-cycle leakage maps* $\Delta_i:\mathcal{S}(\mathcal{H}_{\mathrm{reg}})\to\mathcal{T}(\mathcal{H}_{\mathrm{reg}})$ by

$$\Delta_i(\rho):=\mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}_i(\rho)\Big)-\mathcal{N}_{i,\mathrm{reg}}(\rho),\qquad i=1,2. \tag{45}$$

Then

$$\|\Delta_i(\rho)\|_1\ \le\ \varepsilon^{(i)}_{\mathrm{ps}}\quad\text{for all }\rho\in\mathcal{S}(\mathcal{H}_{\mathrm{reg}}). \tag{46}$$

Step 2: Two-cycle physical and confined evolutions.

Fix an arbitrary initial register state $\rho\in\mathcal{S}(\mathcal{H}_{\mathrm{reg}})$, regarded as a state on $\mathcal{H}_{\mathrm{tot}}$ supported entirely in $\mathcal{H}_{\mathrm{reg}}$.

Define the intermediate physical states

$$\rho^{(1)}:=\mathcal{N}_1(\rho),\qquad\rho^{(2)}:=\mathcal{N}_2(\rho^{(1)})=\mathcal{N}(\rho).$$

Their reduced register states are

$$\mu_1:=\mathrm{Tr}_{\mathrm{comp}}(\rho^{(1)}),\qquad\mu_2:=\mathrm{Tr}_{\mathrm{comp}}(\rho^{(2)})=\mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}_2(\rho^{(1)})\Big).$$

On the other hand, the *confined* register dynamics for two cycles is given by

$$\sigma_1:=\mathcal{N}_{1,\mathrm{reg}}(\rho),\qquad\sigma_2:=\mathcal{N}_{2,\mathrm{reg}}(\sigma_1)=\mathcal{N}_{2,\mathrm{reg}}\Big(\mathcal{N}_{1,\mathrm{reg}}(\rho)\Big).$$

By definition of $\mathcal{N}^{(2\circ1)}_{\mathrm{reg}}$ we have $\sigma_2=\mathcal{N}^{(2\circ1)}_{\mathrm{reg}}(\rho)$.

Thus the two-cycle leakage for input ρ is

$$\mu_2-\sigma_2=\mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}_2(\rho^{(1)})\Big)-\mathcal{N}_{2,\mathrm{reg}}\Big(\mathcal{N}_{1,\mathrm{reg}}(\rho)\Big). \tag{47}$$

Step 3: Decomposition of the two-cycle difference.

We insert and subtract the intermediate confined output $\mathcal{N}_{2,\mathrm{reg}}(\mu_1)$, where μ_1 is the reduced register state after the first physical cycle. Using μ_1 and (45),

$$\begin{aligned}\mu_2-\sigma_2&=\Big[\mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}_2(\rho^{(1)})\Big)-\mathcal{N}_{2,\mathrm{reg}}(\mu_1)\Big]+\Big[\mathcal{N}_{2,\mathrm{reg}}(\mu_1)-\mathcal{N}_{2,\mathrm{reg}}\Big(\mathcal{N}_{1,\mathrm{reg}}(\rho)\Big)\Big]\\&=\Delta_2(\mu_1)+\mathcal{N}_{2,\mathrm{reg}}\Big(\mu_1-\mathcal{N}_{1,\mathrm{reg}}(\rho)\Big),\end{aligned} \tag{48}$$

where in the last step we used the definition of Δ_2 with input $\mu_1\in\mathcal{S}(\mathcal{H}_{\mathrm{reg}})$:

$$\Delta_2(\mu_1)=\mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}_2(\mu_1)\Big)-\mathcal{N}_{2,\mathrm{reg}}(\mu_1).$$

Here we are using the standard convention that, when we apply $\mathcal{N}_2$ to a register state μ_1, we embed μ_1 into $\mathcal{H}_{\mathrm{tot}}$ via the inclusion $\mathcal{H}_{\mathrm{reg}}\hookrightarrow\mathcal{H}_{\mathrm{tot}}$, as in the definition of the single-cycle leakage parameter.

Step 4: Bounding each term and using contractivity.

Taking trace norms in (48) and using the triangle inequality,

$$\|\mu_2 - \sigma_2\|_1 \le \|\Delta_2(\mu_1)\|_1 + \Big\|\mathcal{N}_{2,\mathrm{reg}}\Big(\mu_1 - \mathcal{N}_{1,\mathrm{reg}}(\rho)\Big)\Big\|_1. \tag{49}$$

By (46) with $i = 2$,

$$\|\Delta_2(\mu_1)\|_1 \le \varepsilon_{\mathrm{ps}}^{(2)}.$$

Since $\mathcal{N}_{2,\mathrm{reg}}$ is CPTP, it is trace-norm contractive:

$$\Big\|\mathcal{N}_{2,\mathrm{reg}}\Big(\mu_1 - \mathcal{N}_{1,\mathrm{reg}}(\rho)\Big)\Big\|_1 \le \|\mu_1 - \mathcal{N}_{1,\mathrm{reg}}(\rho)\|_1.$$

But by the definition of Δ_1,

$$\mu_1 - \mathcal{N}_{1,\mathrm{reg}}(\rho) = \Delta_1(\rho),$$

and hence

$$\|\mu_1 - \mathcal{N}_{1,\mathrm{reg}}(\rho)\|_1 = \|\Delta_1(\rho)\|_1 \le \varepsilon_{\mathrm{ps}}^{(1)}.$$

Substituting back into (49) yields

$$\|\mu_2 - \sigma_2\|_1 \le \varepsilon_{\mathrm{ps}}^{(2)} + \varepsilon_{\mathrm{ps}}^{(1)}. \tag{50}$$

Step 5: Taking the supremum over initial register states.

Since the initial state $\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{reg}})$ was arbitrary, taking the supremum over all such ρ in (50) gives

$$\varepsilon_{\mathrm{ps}}^{(2\circ 1)} = \sup_{\rho\in\mathcal{S}(\mathcal{H}_{\mathrm{reg}})} \|\mu_2 - \sigma_2\|_1 \le \varepsilon_{\mathrm{ps}}^{(1)} + \varepsilon_{\mathrm{ps}}^{(2)}.$$

This proves the first claim.

Step 6: Iteration to T identical cycles.

Now let $\mathcal{N}$ be a single fixed CPTP map with leakage parameter $\varepsilon_{\mathrm{ps}}$ and confined channel $\mathcal{N}_{\mathrm{reg}}$. Define

$$\mathcal{N}^{(T)} := \mathcal{N}^T, \qquad \mathcal{N}_{\mathrm{reg}}^{(T)} := \mathcal{N}_{\mathrm{reg}}^T.$$

We prove by induction on $T \in \mathbb{N}$ that

$$\varepsilon_{\mathrm{ps}}^{(T)} := \sup_{\rho\in\mathcal{S}(\mathcal{H}_{\mathrm{reg}})} \Big\|\mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}^{(T)}(\rho)\Big) - \mathcal{N}_{\mathrm{reg}}^{(T)}(\rho)\Big\|_1 \le T\,\varepsilon_{\mathrm{ps}}.$$

Base case $T = 1$*.* This is just the definition of the single-cycle leakage parameter $\varepsilon_{\mathrm{ps}}$.

Induction step. Assume the bound holds for $T-1$. Write the T-cycle map as

$$\mathcal{N}^{(T)} = \mathcal{N} \circ \mathcal{N}^{(T-1)}, \qquad \mathcal{N}_{\mathrm{reg}}^{(T)} = \mathcal{N}_{\mathrm{reg}} \circ \mathcal{N}_{\mathrm{reg}}^{(T-1)}.$$

Applying the two-cycle result we just proved (with $\mathcal{N}_1 := \mathcal{N}^{(T-1)}$ and $\mathcal{N}_2 := \mathcal{N}$, and their associated confined maps $\mathcal{N}_{1,\mathrm{reg}} := \mathcal{N}_{\mathrm{reg}}^{(T-1)}$, $\mathcal{N}_{2,\mathrm{reg}} := \mathcal{N}_{\mathrm{reg}}$) yields

$$\varepsilon_{\mathrm{ps}}^{(T)} \le \varepsilon_{\mathrm{ps}}^{(T-1)} + \varepsilon_{\mathrm{ps}}.$$

By the induction hypothesis, $\varepsilon_{\mathrm{ps}}^{(T-1)} \le (T-1)\varepsilon_{\mathrm{ps}}$, whence

$$\varepsilon_{\mathrm{ps}}^{(T)} \le (T-1)\varepsilon_{\mathrm{ps}} + \varepsilon_{\mathrm{ps}} = T\,\varepsilon_{\mathrm{ps}}.$$

This completes the induction and the proof of the lemma. □

Remark 3.20. Combining Lemma 3.18 and Lemma 3.19, we obtain a uniform per-cycle bound

$$\varepsilon_{\mathrm{ps}} \;\leq\; C_{\dim}\,\eta_{\mathrm{ps}},$$

with $C_{\dim}$ depending only on $\dim \mathcal{H}_{\mathrm{reg}}$ and the fixed local gadget architecture of one QEC cycle, and a composition law

$$\varepsilon_{\mathrm{ps}}^{(T)} \;\leq\; T\,\varepsilon_{\mathrm{ps}} \;\leq\; T\,C_{\dim}\,\eta_{\mathrm{ps}}$$

for T successive applications of the same noisy cycle. Thus, if the off-block Kraus norms η_{ps} are uniformly small for each QEC cycle (with respect to a fixed local architecture), the total leakage from the Twofold register into the complementary (p, s) sectors grows at most linearly in the number of cycles. In particular, for evolutions of length $T \leq \mathrm{poly}(d_L)$, one can ensure that the leakage contribution to the overall error budget remains below any prescribed tolerance by choosing η_{ps} sufficiently small, in direct analogy with the control of local noise parameters in standard threshold theorems for fault-tolerant computation [2, 3, 4, 5].

In later phases we will relate η_{ps} to off-block components of the Fourfold inter-modal correlator $\hat{G}_{\Lambda}$ between the Twofold and (p, s) sectors [8], using the modal bundle and coincident-limit constructions from the emergent-register analysis. This makes both the per-cycle leakage parameter $\varepsilon_{\mathrm{ps}}$ and its architecture-dependent bound $C_{\dim}\eta_{\mathrm{ps}}$ directly computable from the IQR/Tetralemma correlator data.

4 Phase 2: From IQR Correlators to Effective Local Pauli Noise

In this section we specify an *IQR noise class* at the level of the Fourfold Inter-Modal Correlator and derive, under suitable locality and symmetry assumptions, an *effective local Pauli noise model* acting on the emergent Twofold register constructed in Phase 1 (Section 3). This is in direct analogy with the standard local-Pauli noise frameworks used in stabilizer and surface-code fault-tolerance analyses [13, 1, 5, 4].

Throughout we work in the ambient IQR setting of Assumption 2.1 [8], and we restrict attention to a fixed finite patch $\{x_j\}_{j=1}^{n} \subset \mathcal{U} \subset \mathcal{X}_q$ with emergent Twofold register

$$\mathcal{H}_{\mathrm{reg}} = \bigotimes_{j=1}^{n} \mathcal{H}_{(0,1),x_j} \cong (\mathbb{C}^2)^{\otimes n},$$

as in Definition 3.10.

4.1 Ideal vs Perturbed Fourfold Correlator and IQR Noise Class

Definition 4.1 (Ideal and actual Fourfold correlator)**.** Let

$$\hat{G}^{0}_{\Lambda}(\mathfrak{B}_x, \mathfrak{B}_{x'})$$

denote an *ideal* Fourfold Inter-Modal Correlator kernel, corresponding to a reference, “noise-free” dynamics of the PQS continuum in the region of interest, in the sense of the Fourfold Inter-Modal Correlator construction in the IQR/Tetralemma framework [8].

Let

$$\hat{G}_\Lambda(\mathfrak{B}_x, \mathfrak{B}_{x'})$$

denote the *actual* Fourfold correlator kernel measured in the Quantum Holo-State $\omega^{\mathring{\Psi}}$ [8]. We define the *Fourfold noise kernel* by

$$\Delta\hat{G}(\mathfrak{B}_x, \mathfrak{B}_{x'}) := \hat{G}_\Lambda(\mathfrak{B}_x, \mathfrak{B}_{x'}) - \hat{G}^0_\Lambda(\mathfrak{B}_x, \mathfrak{B}_{x'}). \tag{51}$$

Remark 4.2. The ideal correlator $\hat{G}^0_\Lambda$ may be associated, for example, with an idealized QEC cycle in which the emergent register evolves under a purely logical Clifford dynamics (or identity) with no physical errors, in line with standard fault-tolerant constructions at the logical level [13, 2, 3]. The actual correlator includes all environmental and implementation-induced deviations of the IQR correlator away from this ideal sector, as encoded in the Quantum Holo-State [8].

We now introduce a notion of *IQR noise class* in terms of spatial and temporal clustering and smallness on the Twofold block, using the composite correlation distance and temporal structures of the IQR framework.

Assumption 4.3 (Correlation distance and time)**.** *We assume that the IQR framework provides:*

- *a* correlation distance $d_{\mathrm{corr}} : \mathcal{X}_q \times \mathcal{X}_q \to [0, \infty)$*, which is a pseudometric quantifying the relational separation of PQS loci in terms of their correlator-mediated couplings. Concretely,* d_{corr} *is obtained from the operator-valued composite correlation distance* $\hat{d}_{\mathrm{corr}}(x, x')$ *defined in the metric/temporal sector (via Holo-State expectation values or suitable spectral functionals), as in the composite distance constructions of Chapter 5 of [8];*
- *a corresponding* correlation time scale $t_{\mathrm{corr}} > 0$*, arising from the temporal tensor and triadic time displacements in the relational time structure, as developed in the temporal sector of Chapter 5 of [8].*

Definition 4.4 (Twofold block of the correlator)**.** For each pair (x, x'), the Fourfold correlator $\hat{G}_\Lambda(\mathfrak{B}_x, \mathfrak{B}_{x'})$ has components $\hat{G}^{\alpha\beta}_\Lambda(x, x')$ with $\alpha, \beta \in \{0, 1, p, s\}$, where these indices refer to the orthonormal Fourfold basis $\{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$ on each modal fibre constructed in Lemma 3.7 and Definition 3.8.

The *Twofold block* is the 2×2 matrix with entries

$$\hat{G}^{\alpha\beta}_{(0,1)}(x, x') := \hat{G}^{\alpha\beta}_\Lambda(x, x'), \quad \alpha, \beta \in \{0, 1\},$$

and we similarly define the ideal Twofold block $\hat{G}^{0,\alpha\beta}_{(0,1)}(x, x')$ and its noise kernel

$$\Delta\hat{G}_{(0,1)}(x, x') := \hat{G}_{(0,1)}(x, x') - \hat{G}^0_{(0,1)}(x, x').$$

Assumption 4.5 (IQR noise class: clustering and smallness)**.** *We say that the Fourfold noise kernel* $\Delta\hat{G}$ *belongs to the* IQR noise class *if the following conditions hold, mirroring the spatially local and weakly correlated noise assumptions used in threshold theorems [2, 17, 3, 1, 4, 5].*

(i) Spatial locality / clustering.

There exist constants $C > 0$ *and* $\xi > 0$ *(correlation length) such that, for all* $x \neq x'$ *in the region of interest,*

$$\left\|\Delta\hat{G}(\mathfrak{B}_x, \mathfrak{B}_{x'})\right\| \;\le\; C\,\exp\Bigl(-\frac{d_{\mathrm{corr}}(x,x')}{\xi}\Bigr), \tag{52}$$

where $\|\cdot\|$ *is a suitable operator norm on the correlator kernel. This expresses exponential spatial clustering of noise-induced deviations in the Fourfold Inter-Modal Correlator along the emergent correlation distance [8], akin to locality assumptions in topological code analyses [1, 4, 16].*

(ii) Temporal clustering.

The connected time-dependent noise kernel $\Delta\hat{G}(\mathfrak{B}_x, t; \mathfrak{B}_{x'}, t')$ *satisfies an analogous exponential decay in* $|t-t'|/t_{\mathrm{corr}}$*, i.e. there exist constants* $C_t > 0$ *and* $t_{\mathrm{corr}} > 0$ *such that*

$$\left\|\Delta\hat{G}(\mathfrak{B}_x, t; \mathfrak{B}_{x'}, t')\right\| \;\le\; C_t\exp\Bigl(-\frac{|t-t'|}{t_{\mathrm{corr}}}\Bigr). \tag{53}$$

This provides a temporal decorrelation condition along the emergent triadic time structure of the IQR framework [8], analogous to the time-local noise models used in fault-tolerant threshold results [2, 3].

(iii) Smallness in the Twofold sector.

There exists a constant $\epsilon_0 > 0$ *such that the restriction of* $\Delta\hat{G}$ *to the Twofold block obeys the bound*

$$\left\|\Delta\hat{G}_{(0,1)}(\mathfrak{B}_x, \mathfrak{B}_{x'})\right\| \;\le\; \epsilon_0 \qquad \textit{for all } x, x' \textit{ in the patch.} \tag{54}$$

This guarantees that the induced noise on the emergent Twofold sector is perturbatively small, matching the weak-noise regime in which surface-code threshold estimates are typically derived [1, 22, 4, 5].

Remark 4.6. Condition (i) ensures that noise-induced deviations of the Fourfold correlator are exponentially suppressed at large correlation distance in $\mathcal{X}_q$, as measured by the composite correlation distance structure of [8]; (ii) ensures temporal decorrelation beyond the emergent correlation time t_{corr} in the triadic time sector; and (iii) guarantees that the effect of noise on the emergent Twofold sector is uniformly small. These conditions will translate, in subsequent steps, into a microscopic noise model on the register that is local, weakly correlated, and perturbative, placing us within the standard regime of Pauli-noise-based threshold analyses for surface and topological codes [13, 16, 1, 4, 5], but now rooted explicitly in the IQR/Tetralemma correlator data of [8].

4.2 IQR-to-Pauli Reduction on the Twofold Register

We now relate the correlator-level noise kernel $\Delta\hat{G}$ (and in particular its Twofold block $\Delta\hat{G}_{(0,1)}$ from Definition 4.4) to effective channels on the Twofold fibres and derive an approximate *Pauli-diagonal* form for these channels under additional symmetry assumptions, in analogy with standard stabilizer- and surface-code noise modelling in fault-tolerant quantum computation [13, 1, 5, 4].

Local channels and Choi matrices from Twofold correlators

Definition 4.7 (Local Twofold channel from the correlator)**.** Fix a site x_j in the patch. Consider a single QEC cycle with physical CPTP map $\mathcal{N} : \mathcal{S}(\mathcal{H}_{\text{tot}}) \to \mathcal{S}(\mathcal{H}_{\text{tot}})$ on the total Hilbert space, and its restriction to the Twofold register as in Definition 3.14. Denote by $\mathcal{N}_j^{(1)}$ the reduced channel acting on the single-qubit fibre $\mathcal{H}_{(0,1),x_j}$, obtained by:

(a) starting with a product state $\rho_j \otimes \sigma_{\text{rest}}$, where $\rho_j \in \mathcal{S}(\mathcal{H}_{(0,1),x_j})$ and σ_{rest} is some fixed reference state on the other degrees of freedom (including the remaining PQS in the patch, QEC ancillas, and any environment modes, in line with the ambient IQR setting [8]);

(b) applying $\mathcal{N}$;

(c) tracing out all systems except $\mathcal{H}_{(0,1),x_j}$.

This defines a CPTP map

$$\mathcal{N}_j^{(1)} : \mathcal{S}(\mathcal{H}_{(0,1),x_j}) \to \mathcal{S}(\mathcal{H}_{(0,1),x_j}).$$

We call $\mathcal{N}_j^{(1)}$ the *local Twofold channel* at site x_j.

Definition 4.8 (Choi matrix of the local Twofold channel)**.** Let $\Phi^+_{(0,1)}$ denote a maximally entangled state on $\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j}$,

$$|\Phi^+_{(0,1)}\rangle := \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\Big). \tag{55}$$

The *Choi matrix* of $\mathcal{N}_j^{(1)}$ is defined as

$$J(\mathcal{N}_j^{(1)}) := (\mathrm{id} \otimes \mathcal{N}_j^{(1)})\Big(|\Phi^+_{(0,1)}\rangle\langle\Phi^+_{(0,1)}|\Big) \;\in\; \mathcal{B}\Big(\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j}\Big), \tag{56}$$

in the standard Choi–Jamiołkowski channel representation used in quantum information and QEC [5].

Remark 4.9. Because the channel $\mathcal{N}_j^{(1)}$ arises from a correlator-mediated interaction in the IQR framework, the entries of $J(\mathcal{N}_j^{(1)})$ can be expressed as linear combinations of Holo-State expectations of products of Twofold selectors (i.e. the $\{0, 1\}$ corner projectors from the Fourfold selector family) at (possibly) different times [8]. Deviations in the Twofold block $\Delta\hat{G}_{(0,1)}$ thus directly translate into deviations of the Choi matrix from its ideal form.

Lemma 4.10 (Choi deviation vs Twofold noise kernel)**.** *Assume the IQR noise class conditions of Assumption 4.5 for $\Delta\hat{G}_{(0,1)}$, and let $\mathcal{N}_j^{(1)}$ be the local Twofold channel at site x_j. Let $\mathcal{N}_j^{(1),0}$ denote the corresponding ideal local channel constructed from $\hat{G}^0_{(0,1)}$. Then there exists a constant $K > 0$, independent of j in the patch, such that*

$$\Big\| J(\mathcal{N}_j^{(1)}) - J(\mathcal{N}_j^{(1),0}) \Big\|_1 \;\le\; K\,\epsilon_0, \tag{57}$$

where $\|\cdot\|_1$ is the trace norm.

Proof. Fix a site x_j in the patch and a single QEC cycle of duration $[0, T_{\mathrm{cyc}}]$.

Step 1: Correlator-to-channel linearity.

By the IQR / Fourfold-correlator construction of local dynamics (as developed for modal fibres and emergent channels in Chs. 9–10 of [8]), the local Twofold channel $\mathcal{N}_j^{(1)}$ is obtained from the restriction of the Fourfold correlator to the Twofold sector at x_j via a *linear* map. Concretely:

- the QEC gadget for one cycle has a fixed local architecture: a finite schedule of gates, measurements, and couplings between the PQS at x_j and its neighbours/ancillas over a finite set of time intervals within $[0, T_{\mathrm{cyc}}]$;
- in the IQR framework, the effective action of this gadget on the Twofold fibre $\mathcal{H}_{(0,1),x_j}$ can be expressed as a finite sum of time-ordered correlator contributions, with the Twofold block $\hat{G}_{(0,1)}$ entering linearly in the construction of the Choi operator (this is the "correlator-to-channel" step in the GNS/emergent dynamics analysis of [8]).

This can be formalized as follows. Let $\mathcal{K}$ denote the space of operator-valued kernels on $[0, T_{\mathrm{cyc}}]^2$ with values in $\mathcal{B}\Big(\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j}\Big)$, equipped with the norm

$$\|K\|_\infty := \sup_{t,t' \in [0,T_{\mathrm{cyc}}]} \|K(t,t')\|.$$

Then there exists a bounded linear map

$$\mathcal{L}_j : \mathcal{K} \to \mathcal{B}\Big(\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j}\Big)$$

such that

$$J(\mathcal{N}_j^{(1)}) = \mathcal{L}_j\Big[\hat{G}_{(0,1)}(x_j, x_j; \cdot, \cdot)\Big], \qquad J(\mathcal{N}_j^{(1),0}) = \mathcal{L}_j\Big[\hat{G}^0_{(0,1)}(x_j, x_j; \cdot, \cdot)\Big]. \tag{58}$$

The linearity of $\mathcal{L}_j$ is a direct reflection of the fact that the Choi matrix entries are Holo-State expectations of sums and integrals of the Twofold correlators with fixed kernels determined by the QEC gadget, and therefore depend linearly on $\hat{G}_{(0,1)}$.

Step 2: Difference of Choi matrices as image of the noise kernel.

By linearity of $\mathcal{L}_j$ and the definition $\Delta\hat{G}_{(0,1)} := \hat{G}_{(0,1)} - \hat{G}^0_{(0,1)}$, we obtain from (58)

$$J(\mathcal{N}_j^{(1)}) - J(\mathcal{N}_j^{(1),0}) = \mathcal{L}_j\Big[\Delta\hat{G}_{(0,1)}(x_j, x_j; \cdot, \cdot)\Big]. \tag{59}$$

Thus the deviation of the Choi matrices is entirely encoded in the application of the linear functional $\mathcal{L}_j$ to the Twofold noise kernel.

Step 3: Uniform bound on the operator norm of $\mathcal{L}_j$.

By construction, $\mathcal{L}_j$ has the structure of a finite sum of time integrals of the form

$$\mathcal{L}_j[K] = \sum_{\ell=1}^{m} \int_0^{T_{\mathrm{cyc}}} \int_0^{T_{\mathrm{cyc}}} F_j^{(\ell)}(t,t')\, K(t,t')\, G_j^{(\ell)}(t,t')\, \mathrm{d}t\, \mathrm{d}t', \tag{60}$$

for some finite index set $\{\ell = 1, \dots, m\}$, where:

- $F_j^{(\ell)}(t,t')$ and $G_j^{(\ell)}(t,t')$ are bounded operators on $\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j}$ that encode the free/ideal part of the gadget dynamics, local couplings, and any isometries/embeddings used in the construction;
- the number m, the cycle duration T_{cyc}, and the operator norms of the $F_j^{(\ell)}$, $G_j^{(\ell)}$ are determined solely by the fixed local gadget architecture and do not depend on j within the patch.

For any kernel $K \in \mathcal{K}$, using the operator norm on $\mathcal{B}(\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j})$ and the triangle inequality, we have

$$\begin{aligned}
\left\|\mathcal{L}_j[K]\right\| &\le \sum_{\ell=1}^{m} \int_0^{T_{\mathrm{cyc}}}\int_0^{T_{\mathrm{cyc}}} \left\|F_j^{(\ell)}(t,t')\, K(t,t')\, G_j^{(\ell)}(t,t')\right\| \mathrm{d}t\,\mathrm{d}t' \\
&\le \sum_{\ell=1}^{m} \int_0^{T_{\mathrm{cyc}}}\int_0^{T_{\mathrm{cyc}}} \left\|F_j^{(\ell)}(t,t')\right\| \left\|K(t,t')\right\| \left\|G_j^{(\ell)}(t,t')\right\| \mathrm{d}t\,\mathrm{d}t' \\
&\le \Big(\sum_{\ell=1}^{m} \sup_{t,t'} \|F_j^{(\ell)}(t,t')\| \sup_{t,t'} \|G_j^{(\ell)}(t,t')\|\Big) T_{\mathrm{cyc}}^2 \, \|K\|_\infty. \qquad (61)
\end{aligned}$$

Define

$$C_{\mathrm{arch}} := \sum_{\ell=1}^{m} \sup_{t,t'} \|F_j^{(\ell)}(t,t')\| \sup_{t,t'} \|G_j^{(\ell)}(t,t')\|.$$

By homogeneity of the local architecture (each site in the patch sees the same kind of QEC gadget up to relabellings and local unitaries of norm 1), C_{arch} can be chosen independent of j. Then (61) shows that

$$\left\|\mathcal{L}_j[K]\right\| \;\le\; C_{\mathrm{arch}}\, T_{\mathrm{cyc}}^2 \, \|K\|_\infty, \qquad (62)$$

which means that the operator norm of $\mathcal{L}_j$ as a map $(\mathcal{K}, \|\cdot\|_\infty) \to \mathcal{B}(\mathcal{H}_{(0,1),x_j}^{\otimes 2})$ is bounded by

$$\|\mathcal{L}_j\|_{\mathrm{op}} \le C_{\mathrm{arch}}\, T_{\mathrm{cyc}}^2 =: C_0,$$

with $C_0 > 0$ independent of j in the patch.

Step 4: Applying the IQR noise-class smallness bound.

By Assumption 4.5(iii), the restriction of the Fourfold noise kernel to the Twofold block satisfies

$$\left\|\Delta\hat{G}_{(0,1)}(\mathfrak{B}_x, \mathfrak{B}_{x'})\right\| \le \epsilon_0 \qquad \text{for all } x, x' \text{ in the patch.}$$

In particular, for the time-dependent kernel along one site x_j over a single cycle, we have

$$\left\|\Delta\hat{G}_{(0,1)}(x_j, x_j; \cdot, \cdot)\right\|_\infty \le \epsilon_0. \qquad (63)$$

Substituting $K = \Delta\hat{G}_{(0,1)}(x_j, x_j; \cdot, \cdot)$ into (62), and using (59), we obtain

$$\begin{aligned}
\left\|J(\mathcal{N}_j^{(1)}) - J(\mathcal{N}_j^{(1),0})\right\| &= \left\|\mathcal{L}_j\big[\Delta\hat{G}_{(0,1)}(x_j, x_j; \cdot, \cdot)\big]\right\| \\
&\le C_0 \left\|\Delta\hat{G}_{(0,1)}(x_j, x_j; \cdot, \cdot)\right\|_\infty \\
&\le C_0\, \epsilon_0. \qquad (64)
\end{aligned}$$

Step 5: From operator norm to trace norm on the finite Choi space.

Finally, recall that $J(\mathcal{N}_j^{(1)})$ and $J(\mathcal{N}_j^{(1),0})$ act on the two-qubit space $\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j} \cong \mathbb{C}^4$. In any finite-dimensional Hilbert space of dimension d, the operator norm and the trace norm are equivalent, with

$$\|X\|_1 \leq d\,\|X\| \qquad \text{for all } X.$$

Here $d = 4$, so

$$\left\|J(\mathcal{N}_j^{(1)}) - J(\mathcal{N}_j^{(1),0})\right\|_1 \;\leq\; 4\left\|J(\mathcal{N}_j^{(1)}) - J(\mathcal{N}_j^{(1),0})\right\| \;\leq\; 4\,C_0\,\epsilon_0. \tag{65}$$

Setting

$$K := 4\,C_0 = 4\,C_{\text{arch}}\,T_{\text{cyc}}^2$$

gives

$$\left\|J(\mathcal{N}_j^{(1)}) - J(\mathcal{N}_j^{(1),0})\right\|_1 \leq K\,\epsilon_0,$$

with K independent of j in the patch, as required. □

Lemma 4.11 (Diamond-norm deviation of local channels)**.** *Under the hypotheses of Lemma 4.10, there exists a constant $K' > 0$ such that*

$$\left\|\mathcal{N}_j^{(1)} - \mathcal{N}_j^{(1),0}\right\|_\diamond \;\leq\; K'\,\epsilon_0, \tag{66}$$

where $\|\cdot\|_\diamond$ denotes the diamond norm.

Proof. Set

$$\Delta_j := \mathcal{N}_j^{(1)} - \mathcal{N}_j^{(1),0}.$$

This is a linear, Hermiticity-preserving map on $\mathcal{B}(\mathcal{H}_{(0,1),x_j}) \cong \mathcal{B}(\mathbb{C}^2)$.

Step 1: Diamond norm via ancilla and pure states.

By definition, the diamond norm of Δ_j is

$$\|\Delta_j\|_\diamond = \sup_{r\geq 1}\ \sup_{\rho\in\mathcal{S}(\mathbb{C}^r\otimes\mathcal{H}_{(0,1),x_j})} \left\|(\mathrm{id}_r \otimes \Delta_j)(\rho)\right\|_1. \tag{67}$$

For maps on a d-dimensional system, it is standard that the supremum over r may be restricted to $r = d$ (here $d = 2$), and to pure states $\rho = |\psi\rangle\langle\psi|$ on a $d \otimes d$ system (see e.g. standard treatments of the completely bounded trace norm [5]). Thus

$$\|\Delta_j\|_\diamond = \sup_{|\psi\rangle\in\mathcal{H}_{\text{anc}}\otimes\mathcal{H}_{(0,1),x_j}} \left\|(\mathrm{id} \otimes \Delta_j)\big(|\psi\rangle\langle\psi|\big)\right\|_1, \tag{68}$$

where we may take $\dim\mathcal{H}_{\text{anc}} = 2$ and the supremum runs over unit vectors $|\psi\rangle$.

Fix an orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{H}_{(0,1),x_j}$, and let

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}\big(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\big)$$

be the maximally entangled state on $\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j}$ used in the Choi–Jamiołkowski construction.

Any unit vector $|\psi\rangle \in \mathcal{H}_{\mathrm{anc}} \otimes \mathcal{H}_{(0,1),x_j}$ (with both factors of dimension 2) admits a Schmidt decomposition

$$|\psi\rangle = \sum_{k=0}^{1} \lambda_k \, |e_k\rangle \otimes |f_k\rangle, \qquad \lambda_k \geq 0, \quad \sum_k \lambda_k^2 = 1,$$

and can be written as

$$|\psi\rangle = (\mathbb{I} \otimes R)\, |\Phi^+\rangle, \tag{69}$$

for some operator R on $\mathcal{H}_{(0,1),x_j}$ satisfying $\mathrm{Tr}(R^\dagger R) = 2$ (due to the normalization of $|\psi\rangle$) and hence $\|R\| \leq \|R\|_2 = \sqrt{\mathrm{Tr}(R^\dagger R)} = \sqrt{2}$.

Thus every pure ancilla–system input state for the diamond norm can be generated from $|\Phi^+\rangle$ by a fixed operator R acting on the system factor.

Step 2: Expressing $(\mathrm{id} \otimes \Delta_j)(|\psi\rangle\langle\psi|)$ as a linear image of the Choi matrix.

Recall the Choi matrix of Δ_j:

$$J(\Delta_j) = (\mathrm{id} \otimes \Delta_j)\Big(|\Phi^+\rangle\langle\Phi^+|\Big).$$

Using (69) and linearity,

$$\begin{aligned}(\mathrm{id} \otimes \Delta_j)\Big(|\psi\rangle\langle\psi|\Big) &= (\mathrm{id} \otimes \Delta_j)\Big((\mathbb{I} \otimes R)\,|\Phi^+\rangle\langle\Phi^+|\,(\mathbb{I} \otimes R^\dagger)\Big) \\ &= (\mathrm{id} \otimes \Delta_j \circ \mathcal{A}_R)\Big(|\Phi^+\rangle\langle\Phi^+|\Big),\end{aligned} \tag{70}$$

where $\mathcal{A}_R$ is the (completely positive) map $\mathcal{A}_R(X) := RXR^\dagger$.

Define the composed map $\Delta_j^{(R)} := \Delta_j \circ \mathcal{A}_R$. Then (70) reads

$$(\mathrm{id} \otimes \Delta_j)\Big(|\psi\rangle\langle\psi|\Big) = J(\Delta_j^{(R)}),$$

with

$$J(\Delta_j^{(R)}) = (\mathrm{id} \otimes \Delta_j^{(R)})\Big(|\Phi^+\rangle\langle\Phi^+|\Big).$$

By the linearity of the Choi–Jamiołkowski isomorphism, the map $\Delta_j \mapsto J(\Delta_j^{(R)})$ is linear in $J(\Delta_j)$. More explicitly, using the standard inversion formula

$$\Delta_j(X) = 2\,\mathrm{Tr}_1\Big[(X^\mathsf{T} \otimes \mathbb{I})\, J(\Delta_j)\Big] \quad \text{for } X \in \mathcal{B}(\mathbb{C}^2),$$

one checks that there exists a linear map

$$\Gamma_R : \mathcal{B}\Big(\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j}\Big) \to \mathcal{B}\Big(\mathcal{H}_{(0,1),x_j} \otimes \mathcal{H}_{(0,1),x_j}\Big)$$

such that

$$J(\Delta_j^{(R)}) = \Gamma_R\Big(J(\Delta_j)\Big) \tag{71}$$

for all linear maps Δ_j. The construction of Γ_R involves only left/right multiplication by R and the (completely positive, trace-preserving) partial trace, and hence Γ_R itself is completely positive and trace-preserving. In particular, Γ_R is trace-norm contractive:

$$\|\Gamma_R(Y)\|_1 \leq \|Y\|_1 \qquad \text{for all } Y. \tag{72}$$

Combining (70) and (71), we have

$$(\mathrm{id}\otimes\Delta_j)\Big(|\psi\rangle\langle\psi|\Big)=\Gamma_R\Big(J(\Delta_j)\Big). \tag{73}$$

Step 3: Bounding the output trace norm in terms of the Choi trace norm. Using (73) and the contractivity (72), we obtain

$$\Big\|(\mathrm{id}\otimes\Delta_j)\Big(|\psi\rangle\langle\psi|\Big)\Big\|_1=\Big\|\Gamma_R\Big(J(\Delta_j)\Big)\Big\|_1\le\Big\|J(\Delta_j)\Big\|_1. \tag{74}$$

Since this holds for every pure $|\psi\rangle$, taking the supremum over $|\psi\rangle$ in (68) yields

$$\|\Delta_j\|_\diamond\le\Big\|J(\Delta_j)\Big\|_1. \tag{75}$$

Step 4: Applying the Choi deviation bound. By Lemma 4.10, applied to $\mathcal{N}_j^{(1)}$ and $\mathcal{N}_j^{(1),0}$, we have

$$\Big\|J(\mathcal{N}_j^{(1)})-J(\mathcal{N}_j^{(1),0})\Big\|_1\le K\,\epsilon_0$$

for some constant $K>0$ independent of j in the patch. But $J(\Delta_j)=J(\mathcal{N}_j^{(1)}-\mathcal{N}_j^{(1),0})=J(\mathcal{N}_j^{(1)})-J(\mathcal{N}_j^{(1),0})$ by linearity of the Choi map, so

$$\|J(\Delta_j)\|_1\le K\,\epsilon_0.$$

Combining this with (75) gives

$$\|\Delta_j\|_\diamond\le K\,\epsilon_0.$$

Thus the lemma holds with $K':=K>0$, independent of j in the patch:

$$\Big\|\mathcal{N}_j^{(1)}-\mathcal{N}_j^{(1),0}\Big\|_\diamond=\|\Delta_j\|_\diamond\le K'\,\epsilon_0.$$

This completes the proof. □

Approximate Pauli-diagonal form via symmetry

We now add a symmetry assumption ensuring that the ideal local channel is Pauli-diagonal and that noise preserves this property approximately, in line with Pauli-twirled or Pauli-covariant noise models commonly used in stabilizer codes [13, 5].

Definition 4.12 (Local Pauli basis)**.** For each Twofold fibre $\mathcal{H}_{(0,1),x_j}\cong\mathbb{C}^2$ we denote by

$$\{\sigma_0,\sigma_1,\sigma_2,\sigma_3\}:=\{\mathbb{I},X,Y,Z\}$$

the standard Pauli basis of $\mathcal{B}(\mathbb{C}^2)$, with respect to the orthonormal basis $\{|0\rangle_{x_j},|1\rangle_{x_j}\}$.

Assumption 4.13 (Approximate local Pauli covariance)**.** *For each site x_j, the ideal local channel $\mathcal{N}_j^{(1),0}$ is Pauli-diagonal (e.g. the identity channel arising from the ideal PQS dynamics in the absence of noise), and the actual local channel $\mathcal{N}_j^{(1)}$ satisfies an approximate Pauli covariance:*

$$\left\| \mathcal{N}_j^{(1)}(\sigma_\mu) - \sum_{\nu=0}^{3} T_{\mu\nu}^{(j)} \sigma_\nu \right\| \; \leq \; \epsilon_{\text{twirl}}, \tag{76}$$

for all μ, where $T^{(j)}$ is a real 4×4 matrix that is nearly diagonal in the Pauli basis, in the sense that off-diagonal entries $T_{\mu\nu}^{(j)}$, $\mu \neq \nu$, are at most $O(\epsilon_{\text{twirl}})$.

Definition 4.14 (Associated Pauli channel)**.** Given $\mathcal{N}_j^{(1)}$ satisfying Assumption 4.13, define the *associated Pauli channel* Λ_j on $\mathcal{H}_{(0,1),x_j}$ by

$$\Lambda_j(\rho) := \sum_{\mu=0}^{3} q_\mu^{(j)} \, \sigma_\mu \rho \sigma_\mu, \tag{77}$$

where the coefficients $q_\mu^{(j)}$ are determined from the diagonal entries of $T^{(j)}$ (equivalently, they are chosen so that Λ_j has the same action as $\mathcal{N}_j^{(1)}$ on the Pauli-diagonal subalgebra up to $O(\epsilon_{\text{twirl}})$). We define the corresponding *local error rate*

$$p^{(j)} := 1 - q_0^{(j)}. \tag{78}$$

Lemma 4.15 (Local IQR-to-Pauli approximation)**.** *Under Assumptions 4.5 and 4.13, for each site x_j there exists a single-qubit Pauli channel Λ_j such that*

$$\left\| \mathcal{N}_j^{(1)} - \Lambda_j \right\|_\diamond \; \leq \; \epsilon_{\text{loc}}, \tag{79}$$

with

$$\epsilon_{\text{loc}} = O(\epsilon_0 + \epsilon_{\text{twirl}}), \tag{80}$$

and $p^{(j)} = 1 - q_0^{(j)}$ is bounded by some $p > 0$ independent of j in the patch.

Proof. Fix a site x_j. For notational convenience write

$$\mathcal{E}_j := \mathcal{N}_j^{(1)}, \qquad \mathcal{E}_j^0 := \mathcal{N}_j^{(1),0},$$

where $\mathcal{E}_j^0$ is the ideal local channel constructed from $\hat{G}_{(0,1)}^0$.

By Lemma 4.11 (using Lemma 4.10), there exists $C_1 > 0$ such that

$$\|\mathcal{E}_j - \mathcal{E}_j^0\|_\diamond \leq C_1 \, \epsilon_0 \tag{81}$$

for all j in the patch.

Step 1: Pauli transfer matrix representation.

On the Twofold fibre $\mathcal{H}_{(0,1),x_j} \cong \mathbb{C}^2$, let $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} = \{\mathbb{I}, X, Y, Z\}$ denote the Pauli basis (Definition 4.12). Any single-qubit channel $\mathcal{E}$ induces a real 4×4 Pauli transfer matrix $T(\mathcal{E}) = [T_{\mu\nu}]_{\mu,\nu=0}^{3}$ via

$$\mathcal{E}(\sigma_\nu) = \sum_{\mu=0}^{3} T_{\mu\nu} \, \sigma_\mu, \qquad \nu = 0, \ldots, 3. \tag{82}$$

By Assumption 4.13, applied to $\mathcal{E}_j = \mathcal{N}_j^{(1)}$, there exists a real matrix $T^{(j)} = [T^{(j)}_{\mu\nu}]$ such that

$$\Big\|\mathcal{E}_j(\sigma_\mu) - \sum_{\nu=0}^{3} T^{(j)}_{\mu\nu}\,\sigma_\nu\Big\| \le \epsilon_{\mathrm{twirl}} \quad \text{for all } \mu, \tag{83}$$

and $T^{(j)}$ is nearly diagonal: for $\mu \neq \nu$,

$$|T^{(j)}_{\mu\nu}| \le c_{\mathrm{off}}\,\epsilon_{\mathrm{twirl}} \tag{84}$$

for some constant c_{off} independent of j.

Moreover, by construction, $\mathcal{E}_j^0$ is Pauli-diagonal (the ideal dynamics in the Twofold sector is taken to be Clifford/identity in the IQR/Tetralemma framework [8]), so its transfer matrix T_j^0 is diagonal:

$$\mathcal{E}_j^0(\sigma_\mu) = \lambda^{(j)}_\mu\,\sigma_\mu, \qquad \mu = 0,1,2,3, \tag{85}$$

with real eigenvalues $\lambda^{(j)}_\mu$ satisfying the CPTP constraints ($\lambda^{(j)}_0 = 1$, etc.).

Step 2: Relating $T^{(j)}$ and T_j^0 via the diamond bound.

The diamond-distance bound (81) implies that $\mathcal{E}_j$ and $\mathcal{E}_j^0$ act similarly on single-qubit observables. In particular, for each Pauli σ_μ,

$$\Big\|\mathcal{E}_j(\sigma_\mu) - \mathcal{E}_j^0(\sigma_\mu)\Big\|_1 \le 2\,\|\mathcal{E}_j - \mathcal{E}_j^0\|_\diamond \le 2C_1\,\epsilon_0, \tag{86}$$

since $\|\sigma_\mu\|_1 \le 2$ and the diamond norm is the supremum over inputs, including $\sigma_\mu/\|\sigma_\mu\|_1$.

Write

$$\mathcal{E}_j(\sigma_\mu) = \sum_{\nu=0}^{3} T^{(j)}_{\mu\nu}\,\sigma_\nu + R^{(j)}_\mu, \tag{87}$$

where $R^{(j)}_\mu$ is the residual error term from (83), with $\|R^{(j)}_\mu\| \le \epsilon_{\mathrm{twirl}}$. Then

$$\mathcal{E}_j(\sigma_\mu) - \mathcal{E}_j^0(\sigma_\mu) = \sum_{\nu=0}^{3} T^{(j)}_{\mu\nu}\,\sigma_\nu + R^{(j)}_\mu - \lambda^{(j)}_\mu\,\sigma_\mu. \tag{88}$$

Using the orthogonality of the Pauli basis with respect to the Hilbert–Schmidt inner product, and norm equivalences on $\mathcal{B}(\mathbb{C}^2)$, one obtains bounds of the form

$$\Big|T^{(j)}_{\mu\mu} - \lambda^{(j)}_\mu\Big| \le c_1\,\epsilon_0 + c_2\,\epsilon_{\mathrm{twirl}}, \tag{89}$$

and

$$|T^{(j)}_{\mu\nu}| \le c_3\,\epsilon_{\mathrm{twirl}} + c_4\,\epsilon_0 \qquad (\mu \neq \nu), \tag{90}$$

for suitable constants $c_1, \ldots, c_4 > 0$ independent of j.

Step 3: Defining the associated Pauli channel Λ_j.

We now define Λ_j to be the Pauli channel whose action on the Pauli basis coincides with the *diagonal* part of $T^{(j)}$:

$$\Lambda_j(\sigma_\mu) := \tilde{\lambda}^{(j)}_\mu\,\sigma_\mu, \qquad \tilde{\lambda}^{(j)}_\mu := T^{(j)}_{\mu\mu}, \tag{91}$$

for $\mu = 0, 1, 2, 3$. Equivalently, we can parameterize Λ_j as a Pauli channel

$$\Lambda_j(\rho) = \sum_{\mu=0}^{3} q_\mu^{(j)}\, \sigma_\mu \rho \sigma_\mu,$$

with the $q_\mu^{(j)}$ determined by the $\tilde{\lambda}_\mu^{(j)}$ in the usual way.

For single-qubit channels, complete positivity and trace preservation are equivalent to a finite set of linear and quadratic inequalities on the transfer matrix entries. Since T_j^0 is diagonal and corresponds to a CPTP map, and $T^{(j)}$ differs from T_j^0 by at most $O(\epsilon_0 + \epsilon_{\text{twirl}})$ in each entry by (89)–(90), continuity of these constraints implies that for sufficiently small ϵ_0 and ϵ_{twirl} the diagonal entries $\tilde{\lambda}_\mu^{(j)}$ can be chosen so that Λ_j is CPTP and

$$\|\Lambda_j - \mathcal{E}_j^0\|_\diamond \le C_2\, \epsilon_0 + C_3\, \epsilon_{\text{twirl}} \tag{92}$$

for some constants $C_2, C_3 > 0$ independent of j. (This uses that, for single-qubit channels, the diamond norm is equivalent to any reasonable matrix norm on the transfer matrix.)

Step 4: Bounding $\|\mathcal{E}_j - \Lambda_j\|_\diamond$.

Using the triangle inequality in diamond norm,

$$\begin{aligned} \|\mathcal{E}_j - \Lambda_j\|_\diamond &\le \|\mathcal{E}_j - \mathcal{E}_j^0\|_\diamond + \|\mathcal{E}_j^0 - \Lambda_j\|_\diamond \\ &\le C_1\, \epsilon_0 + C_2\, \epsilon_0 + C_3\, \epsilon_{\text{twirl}} \\ &= C_4\, \epsilon_0 + C_3\, \epsilon_{\text{twirl}}, \end{aligned} \tag{93}$$

for some $C_4 > 0$ independent of j. Thus, setting

$$\epsilon_{\text{loc}} := C_4\, \epsilon_0 + C_3\, \epsilon_{\text{twirl}},$$

we obtain

$$\|\mathcal{N}_j^{(1)} - \Lambda_j\|_\diamond = \|\mathcal{E}_j - \Lambda_j\|_\diamond \le \epsilon_{\text{loc}},$$

with $\epsilon_{\text{loc}} = O(\epsilon_0 + \epsilon_{\text{twirl}})$ as claimed.

Step 5: Uniform bound on the local error rate $p^{(j)}$.

By definition,

$$p^{(j)} := 1 - q_0^{(j)}$$

is the total probability of non-identity Pauli errors in Λ_j. For the ideal channel $\mathcal{E}_j^0$ we have $q_0^{(j),0} = 1$ and $q_\mu^{(j),0} = 0$ for $\mu \ge 1$ (no errors), so $p^{(j),0} = 0$. Since Λ_j is within diamond distance ϵ_{loc} of $\mathcal{E}_j$, and $\mathcal{E}_j$ is within diamond distance $C_1\epsilon_0$ of $\mathcal{E}_j^0$, the eigenvalues of $T(\Lambda_j)$ (and hence the error probability $p^{(j)}$) are small perturbations of those of $T(\mathcal{E}_j^0)$. In particular, there exists a constant $c_p > 0$ such that

$$p^{(j)} \le c_p\, (\epsilon_0 + \epsilon_{\text{twirl}}) =: p, \tag{94}$$

for all j in the patch.

The resulting uniform bound $p^{(j)} \le p$ places the effective local noise in the same small-p, local-Pauli regime that underlies standard fault-tolerance threshold theorems for stabilizer and surface-code architectures [2, 3, 1, 4, 5]. This completes the proof. □

Multi-site correlations and approximate product structure

We finally address correlations between different sites in the patch and show that, under exponential clustering of the IQR noise kernel, the global noise channel is close to a product of the local Pauli channels Λ_j plus a small correlated tail, as in weakly correlated local noise models for topological codes [1, 16, 4, 5].

Definition 4.16 (Global register channel for one cycle)**.** Let $\mathcal{N} : \mathcal{S}(\mathcal{H}_{\text{tot}}) \to \mathcal{S}(\mathcal{H}_{\text{tot}})$ be the CPTP map describing one QEC cycle, and let $\mathcal{N}_{\text{reg}}$ be the confined register channel as in Definition 3.14. The induced *global register channel* on $\mathcal{H}_{\text{reg}} = \bigotimes_{j=1}^{n} \mathcal{H}_{(0,1),x_j}$ is

$$\mathcal{N}^{(n)} : \mathcal{S}(\mathcal{H}_{\text{reg}}) \to \mathcal{S}(\mathcal{H}_{\text{reg}}), \qquad \mathcal{N}^{(n)}(\rho) := \mathcal{N}_{\text{reg}}(\rho).$$

Lemma 4.17 (Exponential clustering and multi-site Kraus structure)**.** *Assume the spatial clustering condition of Assumption 4.5, and let $\mathcal{N}^{(n)}$ be the global register channel. Then $\mathcal{N}^{(n)}$ admits a Kraus decomposition*

$$\mathcal{N}^{(n)}(\rho) = \sum_{\alpha} K_\alpha \rho K_\alpha^{\dagger}, \tag{95}$$

such that each Kraus operator K_α can be written as a sum of terms supported on clusters of sites of bounded diameter, and the norm of terms acting nontrivially on widely separated sites is exponentially suppressed in the cluster diameter with rate set by ξ.

Sketch of proof. By construction in the IQR framework, the one-cycle evolution on $\mathcal{H}_{\text{tot}}$ is generated by correlator-mediated interactions of PQS loci encoded in the Fourfold kernel $\hat{G}_\Lambda(\mathfrak{B}_x, \mathfrak{B}_{x'})$ and its time-dependent version [8]. At the level of the emergent Twofold register, the effective dynamics over a single QEC cycle can be expressed as a time-ordered exponential (Dyson series) of an effective interaction generator whose coefficients are given by connected correlators of selectors supported on finite subsets of sites in the patch.

More concretely, one can write the Stinespring dilation of $\mathcal{N}$ as

$$\mathcal{N}(\cdot) = \mathrm{Tr}_{\text{env}}\Big[U_{\text{eff}}\,(\cdot \otimes \sigma_{\text{env}})\,U_{\text{eff}}^{\dagger}\Big],$$

where the effective unitary U_{eff} is a time-ordered exponential built from a sum of interaction terms $H_C(t)$ supported on finite clusters $C \subset \{1, \ldots, n\}$ of PQS loci, with operator norms determined by multi-point Fourfold correlators on C. The exponential clustering of the noise kernel $\Delta\hat{G}$ from Assumption 4.5(i) implies that the connected part of any $|C|$-point correlator decays exponentially with the correlation diameter $\mathrm{diam}_{\text{corr}}(C)$ (measured using d_{corr}), i.e.

$$\Big\|\mathrm{conn}\,\Delta\hat{G}_C\Big\| \lesssim \exp\Big(-\frac{\mathrm{diam}_{\text{corr}}(C)}{\xi}\Big),$$

in the sense of the cluster decompositions developed in the IQR setting for PQS continua [8, Chs. 9–10].

Expanding U_{eff} in a Dyson or cumulant series and regrouping terms by their spatial support yields an effective Kraus representation

$$\mathcal{N}^{(n)}(\rho) = \sum_{\alpha} K_\alpha \rho K_\alpha^{\dagger},$$

where each Kraus operator K_α is a sum over words supported on a finite cluster $C(\alpha) \subset \{1, \ldots, n\}$:

$$K_\alpha = \sum_{C(\alpha)} K_{\alpha, C(\alpha)}.$$

The operator norm of $K_{\alpha, C(\alpha)}$ is bounded by a product of coefficients involving connected multi-point correlators on $C(\alpha)$; the exponential clustering of these correlators implies

$$\|K_{\alpha, C(\alpha)}\| \lesssim \exp\Big(-\frac{\mathrm{diam}_{\mathrm{corr}}(C(\alpha))}{\xi}\Big),$$

up to combinatorial factors that are bounded uniformly in the patch. Hence terms whose support $C(\alpha)$ has large correlation diameter contribute with exponentially small norm, while dominant contributions come from clusters of bounded diameter (few-body noise).

Tracing out $\mathcal{H}_{\mathrm{comp}}$ to obtain $\mathcal{N}_{\mathrm{reg}}$ does not increase operator norms and preserves the same cluster support structure on $\mathcal{H}_{\mathrm{reg}}$, so the above Kraus decomposition carries over to $\mathcal{N}^{(n)}$. This is the standard cluster-expansion picture for channels generated by finite-range or rapidly decaying interactions in lattice quantum systems, now instantiated in the IQR / Fourfold correlator context [1, 16, 4, 5]. □

Lemma 4.18 (Approximate tensor-product decomposition)**.** *Let Λ_j be the local Pauli channels from Lemma 4.15, and let $\Lambda^{(n)} := \bigotimes_{j=1}^{n} \Lambda_j$ denote their tensor product on $\mathcal{H}_{\mathrm{reg}}$. Under the assumptions of Lemma 4.17, there exists a CPTP map $\mathcal{N}^{(n)}_{\mathrm{corr}}$ such that*

$$\mathcal{N}^{(n)} = \Lambda^{(n)} + \mathcal{N}^{(n)}_{\mathrm{corr}}, \tag{96}$$

and

$$\left\|\mathcal{N}^{(n)}_{\mathrm{corr}}\right\|_\diamond \leq \epsilon_{\mathrm{corr}}(n) \tag{97}$$

with

$$\epsilon_{\mathrm{corr}}(n) = O\Big(n\, e^{-r/\xi}\Big), \tag{98}$$

where r is a minimal separation scale beyond nearest-neighbour interactions in the patch geometry.

Sketch of proof. Let $\mathcal{N}^{(n)}$ have the clustered Kraus decomposition from Lemma 4.17. For each Kraus label α, let $C(\alpha) \subset \{1, \ldots, n\}$ denote the minimal cluster of sites on which K_α acts nontrivially.

Fix a geometric scale $r > 0$ such that any cluster C with $\mathrm{diam}_{\mathrm{corr}}(C) > r$ necessarily involves sites beyond nearest-neighbour range in the patch (e.g. on a 2D local-lattice embedding of the patch). Partition the Kraus operators into

$$\mathcal{I}_{\mathrm{small}} := \{\alpha : \mathrm{diam}_{\mathrm{corr}}(C(\alpha)) \leq r\}, \qquad \mathcal{I}_{\mathrm{large}} := \{\alpha : \mathrm{diam}_{\mathrm{corr}}(C(\alpha)) > r\}.$$

Define two CP maps

$$\mathcal{N}^{(n)}_{\mathrm{small}}(\rho) := \sum_{\alpha \in \mathcal{I}_{\mathrm{small}}} K_\alpha \rho K_\alpha^\dagger, \tag{99}$$

$$\mathcal{N}^{(n)}_{\mathrm{large}}(\rho) := \sum_{\alpha \in \mathcal{I}_{\mathrm{large}}} K_\alpha \rho K_\alpha^\dagger. \tag{100}$$

Then $\mathcal{N}^{(n)} = \mathcal{N}^{(n)}_{\text{small}} + \mathcal{N}^{(n)}_{\text{large}}$. We will (i) approximate $\mathcal{N}^{(n)}_{\text{small}}$ by the product $\Lambda^{(n)}$, and (ii) bound the diamond norm of $\mathcal{N}^{(n)}_{\text{large}}$.

(i) Matching the local marginals. By construction, each Λ_j approximates the local marginal channel $\mathcal{N}^{(1)}_j$ on site x_j in diamond norm (Lemma 4.15). Using the small-cluster Kraus operators $\{K_\alpha\}_{\alpha\in\mathcal{I}_{\text{small}}}$, one can adjust the parameters of Λ_j so that the one-site marginals of $\mathcal{N}^{(n)}_{\text{small}}$ match those of $\Lambda^{(n)}$ up to errors of order $O(\epsilon_{\text{loc}})$, and, if desired, also partially absorb nearest-neighbour two-site contributions into a redefinition of the local rates, in the standard way used in local-noise threshold analyses [2, 17, 3, 1, 4, 5]. The resulting difference $\mathcal{N}^{(n)}_{\text{small}} - \Lambda^{(n)}$ is a CPTP map whose Kraus operators are supported on clusters of bounded diameter (nearest neighbours and possibly small plaquettes) and whose coefficients are $O(\epsilon_{\text{loc}})$; for a fixed patch geometry this contribution can be absorbed into the definition of the local error parameters and does not create long-range correlations.

(ii) Bounding the large-cluster correlated tail. Define

$$\mathcal{N}^{(n)}_{\text{corr}} := \mathcal{N}^{(n)}_{\text{large}} + \Big(\mathcal{N}^{(n)}_{\text{small}} - \Lambda^{(n)}\Big).$$

We now bound its diamond norm. For a CP map with Kraus operators $\{L_\beta\}$, one has the bound

$$\|\Phi\|_\diamond \;\le\; 2\sum_\beta \|L_\beta\|^2,$$

up to a fixed dimension-dependent constant, since the Choi matrix of Φ has trace norm controlled by the sum of $\|L_\beta\|^2$. Applying this to $\mathcal{N}^{(n)}_{\text{large}}$ and using the exponential bound on $\|K_{\alpha,C(\alpha)}\|$ from Lemma 4.17, we find

$$\Big\|\mathcal{N}^{(n)}_{\text{large}}\Big\|_\diamond \;\lesssim\; \sum_{\alpha\in\mathcal{I}_{\text{large}}} \exp\!\Big(-2\frac{\operatorname{diam}_{\text{corr}}(C(\alpha))}{\xi}\Big).$$

The number of possible large clusters $C(\alpha)$ of diameter in $[r, r+\Delta r)$ that intersect the patch is bounded by a factor of order $O(n)$ times a geometry-dependent constant, since each such cluster must contain at least one site of the patch and the local connectivity is bounded. Summing the exponentially decaying contributions over cluster diameters $\ge r$ therefore yields a bound of the form

$$\Big\|\mathcal{N}^{(n)}_{\text{large}}\Big\|_\diamond \le c_1\, n\, e^{-r/\xi},$$

for some constant $c_1 > 0$ depending only on the local geometry and the correlator bounds.

The term $\mathcal{N}^{(n)}_{\text{small}} - \Lambda^{(n)}$ contributes at most $O(\epsilon_{\text{loc}})$ in diamond norm, which can be absorbed into the local-noise parameters $p^{(j)}$ already appearing in the threshold analysis. Collecting both contributions, we obtain

$$\Big\|\mathcal{N}^{(n)}_{\text{corr}}\Big\|_\diamond \;\le\; c_1\, n\, e^{-r/\xi} + c_2\,\epsilon_{\text{loc}} \;=\; \epsilon_{\text{corr}}(n),$$

for some constants $c_1, c_2 > 0$. In the regime where $\epsilon_{\text{loc}} = O(\epsilon_0 + \epsilon_{\text{twirl}})$ is small and $r \gg \xi$, the $n\, e^{-r/\xi}$ term dominates the large-scale behaviour and we may summarize this as

$$\epsilon_{\text{corr}}(n) = O\Big(n\, e^{-r/\xi}\Big),$$

as claimed. This realizes the standard "weakly correlated local noise" structure required in threshold theorems for topological and surface codes, here derived directly from the IQR/Fourfold correlator clustering [1, 16, 4, 5]. □

IQR-to-Pauli Lemma and Local Noise Regime

We can now state the central reduction lemma of Phase 2, which places the IQR noise description into the familiar local-Pauli-noise framework used in stabilizer and surface-code threshold analyses [13, 1, 5, 4], realizing at the level of the emergent Twofold register the heuristic IQR-to-Pauli reduction described in [8, Chs. 9–10].

Lemma 4.19 (IQR $\Rightarrow$ effective local Pauli noise)**.** *Let the Fourfold noise kernel $\Delta\hat{G}$ satisfy the IQR noise class conditions of Assumption 4.5, and assume approximate local Pauli covariance as in Assumption 4.13. Consider a single QEC cycle with global register channel $\mathcal{N}^{(n)}$ on $\mathcal{H}_{\mathrm{reg}}$ (Definition 4.16) [5, 8].*

Then for each site x_j there exists a single-qubit Pauli channel

$$\Lambda_j(\rho) = \sum_{P\in\{\mathbb{I},X,Y,Z\}} q_P^{(j)} P\rho P,$$

with local error rate $p^{(j)} := 1 - q_{\mathbb{I}}^{(j)} \leq p$ for some $p > 0$ independent of j, such that the following hold:

*(a) (*Local approximation.*) The local Twofold channel $\mathcal{N}_j^{(1)}$ satisfies*

$$\left\|\mathcal{N}_j^{(1)} - \Lambda_j\right\|_\diamond \leq \epsilon_{\mathrm{loc}}, \tag{101}$$

with $\epsilon_{\mathrm{loc}} = O(\epsilon_0 + \epsilon_{\mathrm{twirl}})$, as in the local IQR-to-Pauli approximation of Lemma 4.15.

*(b) (*Global approximate product structure.*) The global register channel can be decomposed as*

$$\mathcal{N}^{(n)} = \Lambda^{(n)} + \mathcal{N}_{\mathrm{corr}}^{(n)}, \qquad \Lambda^{(n)} := \bigotimes_{j=1}^{n} \Lambda_j, \tag{102}$$

with

$$\left\|\mathcal{N}_{\mathrm{corr}}^{(n)}\right\|_\diamond \leq \epsilon_{\mathrm{corr}}(n), \tag{103}$$

where $\epsilon_{\mathrm{corr}}(n) = O\Big(n\, e^{-r/\xi}\Big)$ for a suitable separation scale r, reflecting the exponential clustering of correlations in the IQR noise class [1, 16, 4, 5, 8].

Proof. For part (a), we invoke the local IQR-to-Pauli approximation (Lemma 4.15). Under Assumption 4.5, the Twofold noise kernel $\Delta\hat{G}_{(0,1)}$ is uniformly small in operator norm (parameter ϵ_0) and exhibits temporal clustering, so Lemma 4.10 bounds the trace-norm deviation of the Choi matrix of $\mathcal{N}_j^{(1)}$ from its ideal Pauli-diagonal counterpart in terms of ϵ_0. Lemma 4.11 then upgrades this to a diamond-norm deviation of order $O(\epsilon_0)$.

Assumption 4.13 guarantees that $\mathcal{N}_j^{(1)}$ is approximately Pauli-covariant, so that its off-diagonal components in the Pauli basis are of order $\epsilon_{\mathrm{twirl}}$. Projecting onto the Pauli-diagonal part defines a single-qubit Pauli channel Λ_j whose Pauli eigenvalues differ from those of $\mathcal{N}_j^{(1)}$ by at most $O(\epsilon_{\mathrm{twirl}})$. A standard norm estimate comparing a channel to its Pauli-twirled version (cf. [13, 5]) then yields

$$\left\|\mathcal{N}_j^{(1)} - \Lambda_j\right\|_\diamond \leq c_1\epsilon_0 + c_2\epsilon_{\mathrm{twirl}},$$

for constants $c_1, c_2 > 0$ independent of j, which we summarize as $\epsilon_{\mathrm{loc}} = O(\epsilon_0 + \epsilon_{\mathrm{twirl}})$. The bound $p^{(j)} \leq p$ for some $p > 0$ independent of j follows from the fact that $\mathcal{N}_j^{(1)}$ is a small perturbation of the ideal (noiseless) channel and positivity / trace-preservation constrain the Pauli eigenvalues to lie in $[0, 1]$; in particular, $q_{\mathbb{I}}^{(j)}$ is close to 1 and hence $p^{(j)}$ is uniformly small in the weak-noise regime, as in the local-noise assumptions of surface-code threshold theorems [2, 3, 1, 4, 5].

For part (b), we use the multi-site structure provided by the IQR noise class. By Lemma 4.17, the global register channel $\mathcal{N}^{(n)}$ admits a Kraus decomposition with Kraus operators supported on spatial clusters $C \subset \{1, \ldots, n\}$, and the operator norm of the contribution supported on C is exponentially suppressed in the correlation diameter $\mathrm{diam}_{\mathrm{corr}}(C)$ with decay length ξ.

We now split the Kraus operators into those supported on small clusters $\mathrm{diam}_{\mathrm{corr}}(C) \leq r$ (nearest-neighbour and finite range) and those with $\mathrm{diam}_{\mathrm{corr}}(C) > r$, where r is chosen larger than the minimal separation scale beyond nearest neighbours in the patch geometry. The small-cluster contributions renormalize the local channels Λ_j (and, if desired, nearest-neighbour two-site terms) but do not generate long-range correlations; the resulting channel can be arranged to match the marginals of $\Lambda^{(n)} := \bigotimes_{j=1}^{n} \Lambda_j$ up to diamond-norm deviations of order $O(\epsilon_{\mathrm{loc}})$, which are already absorbed into the local error parameters.

The large-cluster contributions define a CPTP map $\mathcal{N}_{\mathrm{corr}}^{(n)}$ whose Kraus operators have support diameter $> r$ and norm bounded by $\exp(-\,\mathrm{diam}_{\mathrm{corr}}/\xi)$. The diamond norm of $\mathcal{N}_{\mathrm{corr}}^{(n)}$ can be bounded in terms of the sum of squared Kraus norms; counting possible cluster positions and using the exponential decay yields

$$\left\| \mathcal{N}_{\mathrm{corr}}^{(n)} \right\|_{\diamond} \leq c\, n\, e^{-r/\xi},$$

for some constant $c > 0$ depending only on the local geometry and correlator bounds. This is precisely the behaviour $\epsilon_{\mathrm{corr}}(n) = O(ne^{-r/\xi})$ claimed in the statement, in direct analogy with weakly correlated local noise models used in topological and surface-code analyses [1, 16, 4, 5]. Combining the local approximation of part (a) with this cluster expansion completes the proof. □

Corollary 4.20 (Local Pauli noise regime on the emergent register)**.** *Under the hypotheses of Lemma 4.19 and the leakage control results of Phase 1 (in particular Definition 3.15 and Lemmas 3.18–3.19), the emergent Twofold register $\mathcal{H}_{\mathrm{reg}}$ is subject, during one QEC cycle, to an effective noise model of the form*

$$\mathcal{N}_{\mathrm{eff}}^{(n)} = \left(\bigotimes_{j=1}^{n} \Lambda_j \right) + \mathcal{N}_{\mathrm{corr}}^{(n)} + \mathcal{N}_{\mathrm{leak}}^{(n)}, \tag{104}$$

where:

- *each Λ_j is a single-qubit Pauli channel with error rate $p^{(j)} \leq p$, fitting the standard local-Pauli noise paradigm for stabilizer codes [13, 5];*
- *$\|\mathcal{N}_{\mathrm{corr}}^{(n)}\|_{\diamond} \leq \epsilon_{\mathrm{corr}}(n)$ with exponential suppression in the correlation distance (as in Lemma 4.19 and the clustering assumptions of Assumption 4.5);*

- $\mathcal{N}_{\text{leak}}^{(n)}$ *collects contributions from leakage into* (p, s) *sectors, defined by*

  $$\mathcal{N}_{\text{leak}}^{(n)}(\rho) := \mathrm{Tr}_{\text{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}^{(n)}(\rho),$$

  so that $\mathrm{Tr}_{\text{comp}}\Big(\mathcal{N}(\rho)\Big) = \mathcal{N}^{(n)}(\rho) + \mathcal{N}_{\text{leak}}^{(n)}(\rho)$ *for all* $\rho \in \mathcal{S}(\mathcal{H}_{\text{reg}})$*. By Definition 3.15,*

  $$\sup_{\rho\in\mathcal{S}(\mathcal{H}_{\text{reg}})} \left\|\mathcal{N}_{\text{leak}}^{(n)}(\rho)\right\|_1 = \varepsilon_{\text{ps}},$$

  and in finite dimensions this implies $\|\mathcal{N}_{\text{leak}}^{(n)}\|_\diamond = O(\varepsilon_{\text{ps}})$*. Over* T *cycles, Lemma 3.19 yields* $\varepsilon_{\text{ps}}^{(T)} \le T\,\varepsilon_{\text{ps}}$*, so the leakage contribution grows at most linearly in* T*.*

Thus, up to controlled corrections of order $O(\epsilon_0 + \epsilon_{\text{twirl}} + \varepsilon_{\text{ps}})$ *and an exponentially small correlated tail, the noise acting on the emergent qubit lattice is equivalent to standard* local Pauli noise *with bounded error rate* p*, placing the IQR-derived model squarely within the regime of known surface-code threshold theorems [2, 17, 3, 1, 22, 4, 5, 8].*

Remark 4.21. Corollary 4.20 is the desired endpoint of Phase 2: it realizes, in precise operator-algebraic form, the reduction of IQR/Fourfold correlator noise on a PQS continuum to a local Pauli noise model on the emergent Twofold register, with weakly correlated tails and controlled leakage, as anticipated in [8, Chs. 9–10]. This places the IQR-based emergent register squarely within the class of noise models for which surface-code threshold theorems are known to hold (local, weakly correlated Pauli noise plus small leakage) [1, 22, 4, 5]. Subsequent phases will use this structure to derive quantitative bounds on logical error rates $p_L^{(L)}$ and holosymmetric process valuations $v_{\text{Holo}}^{(L,T)}$ as functions of the code distance and the physical noise parameters $(p, \delta, \varepsilon_{\text{ps}})$, in direct analogy with existing threshold analyses for concatenated and topological codes [2, 3, 5, 8].

4.3 Target Threshold Claim in Valuation Language

We now restate the threshold claim of Section 2 in a form that makes the dependence on the IQR noise parameters and the Phase 1–2 reductions fully explicit. In particular, we connect the noise parameters $(p, \delta, \varepsilon_{\text{ps}})$ of Definition 2.18 to the effective local Pauli noise model of Corollary 4.20, and then invoke standard surface-code threshold theorems to control the logical Thesis valuation and the process-level holosymmetric valuation [2, 17, 3, 1, 22, 4, 5].

Recall that:

- the per-PQS physical error rate p is defined as the supremum of the single-qubit Pauli error rates $p^{(j)}$ appearing in the local Pauli channels Λ_j of Corollary 4.20;
- the leakage parameter ε_{ps} is defined in Definition 3.15 and controlled under composition by Lemma 3.19;
- the differential noise parameter δ captures the discrepancy between the A and B histories at the level of their local Pauli noise distributions (Definition 2.18 and Definition 2.11).

Logical sector: from effective Pauli noise to $p_L^{(L)}$

We first isolate the dependence of the logical error probability $p_L^{(L)}$ on the IQR-derived noise parameters and the code distance.

Assumption 4.22 (Surface-code threshold for local Pauli noise)**.** *Consider a family of distance-d_L surface-code patches $\{\mathcal{C}_L\}_L$ as in Definition 2.3, with computational gadgets built from bounded-depth Clifford/non-Clifford circuits and local syndrome extraction, as in standard 2D surface-code architectures [16, 1, 4, 5].*

Assume that for any local Pauli noise model of the form

$$\Lambda^{(n_L)} = \bigotimes_{j=1}^{n_L} \Lambda_j,$$

with per-site Pauli error rate $p^{(j)} \leq p < p_*$ for some threshold constant $p_* > 0$ independent of L, the logical error probability for one QEC cycle satisfies a bound

$$p_{L,\text{Pauli}}^{(L)}(p) \;\leq\; f_{\text{th}}(d_L, p), \tag{105}$$

where f_{th} is a decreasing function of d_L whenever $p < p_*$, typically of the form $f_{\text{th}}(d_L, p) \leq A\exp(-\gamma d_L)$ or $f_{\text{th}}(d_L, p) \leq A(p/p_*)^{\kappa d_L}$ for some constants $A, \gamma, \kappa > 0$, as in [1, 22, 4, 5].

Definition 4.23 (Register-level non-Pauli deviation per cycle)**.** Let $\mathcal{N}^{(n_L)}$ be the global register channel on $\mathcal{H}_{\text{reg}}$ for one QEC cycle (Definition 4.16), and let $\Lambda^{(n_L)} := \bigotimes_{j=1}^{n_L} \Lambda_j$ be the product of local Pauli channels supplied by Lemma 4.19.

Define the *non-Pauli deviation per cycle* to be

$$\Delta_{\text{NP}}^{(L)} := \left\| \mathcal{N}^{(n_L)} - \Lambda^{(n_L)} \right\|_\diamond. \tag{106}$$

By Corollary 4.20, we may bound

$$\Delta_{\text{NP}}^{(L)} \;\leq\; C_{\text{loc}}\Big(\epsilon_0 + \epsilon_{\text{twirl}}\Big) + \epsilon_{\text{corr}}(n_L) + C_{\text{leak}}\varepsilon_{\text{ps}}, \tag{107}$$

for some constants $C_{\text{loc}}, C_{\text{leak}} > 0$ depending only on the local gadget architecture and the finite ancilla/environment dimension, and with $\epsilon_{\text{corr}}(n_L) = O(n_L e^{-r/\xi})$ as in Lemma 4.18.

Lemma 4.24 (Logical error under IQR-derived noise)**.** *Let $p_L^{(L)}$ be the worst-case logical error probability at size L (Definition 2.7) for the QEC cycle driven by $\mathcal{N}^{(n_L)}$, and let $p_{L,\text{Pauli}}^{(L)}(p)$ be the logical error probability for the same code and gadget schedule under the ideal local Pauli noise model $\Lambda^{(n_L)}$ with per-site rate p.*

Then

$$p_L^{(L)} \;\leq\; p_{L,\text{Pauli}}^{(L)}(p) + \Delta_{\text{NP}}^{(L)}. \tag{108}$$

Proof. Let $\mathcal{N}_{\text{log}}^{(L)}$ and $\mathcal{N}_{\text{log,Pauli}}^{(L)}$ denote the logical channels induced by $\mathcal{N}^{(n_L)}$ and $\Lambda^{(n_L)}$, respectively, via the encoding / decoding procedure of Definition 2.5. Because encoding and decoding are isometries followed by partial traces, they are completely positive trace-preserving maps and hence contractive in diamond norm. Thus

$$\left\| \mathcal{N}_{\text{log}}^{(L)} - \mathcal{N}_{\text{log,Pauli}}^{(L)} \right\|_\diamond \;\leq\; \left\| \mathcal{N}^{(n_L)} - \Lambda^{(n_L)} \right\|_\diamond = \Delta_{\text{NP}}^{(L)}. \tag{109}$$

For any input logical state $\rho_{\text{log}} \in \mathcal{S}(\mathcal{C}_L)$, the trace distance between the outputs of the two logical channels is bounded by the diamond norm:

$$\left\| \mathcal{N}^{(L)}_{\text{log}}(\rho_{\text{log}}) - \mathcal{N}^{(L)}_{\text{log,Pauli}}(\rho_{\text{log}}) \right\|_1 \leq \Delta^{(L)}_{\text{NP}}.$$

By definition, $p^{(L)}_{L,\text{Pauli}}(p)$ is the supremum, over input logical states, of the distinguishing advantage between $\mathcal{N}^{(L)}_{\text{log,Pauli}}$ and the ideal logical channel $\mathcal{N}^{(L)}_{\text{ideal}}$, while $p^{(L)}_L$ is the corresponding supremum for $\mathcal{N}^{(L)}_{\text{log}}$. Applying the triangle inequality for the trace norm and maximizing over inputs yields

$$\begin{aligned} p^{(L)}_L &= \sup_{\rho_{\text{log}}} \tfrac{1}{2} \left\| \mathcal{N}^{(L)}_{\text{log}}(\rho_{\text{log}}) - \mathcal{N}^{(L)}_{\text{ideal}}(\rho_{\text{log}}) \right\|_1 \\ &\leq \sup_{\rho_{\text{log}}} \tfrac{1}{2} \left\| \mathcal{N}^{(L)}_{\text{log,Pauli}}(\rho_{\text{log}}) - \mathcal{N}^{(L)}_{\text{ideal}}(\rho_{\text{log}}) \right\|_1 + \sup_{\rho_{\text{log}}} \tfrac{1}{2} \left\| \mathcal{N}^{(L)}_{\text{log}}(\rho_{\text{log}}) - \mathcal{N}^{(L)}_{\text{log,Pauli}}(\rho_{\text{log}}) \right\|_1 \\ &\leq p^{(L)}_{L,\text{Pauli}}(p) + \tfrac{1}{2} \Delta^{(L)}_{\text{NP}}. \end{aligned}$$

Since $\Delta^{(L)}_{\text{NP}} \leq 2$ for any CPTP maps, we may absorb the factor $\frac{1}{2}$ into a redefinition of the constant in Definition 4.23, and we write the bound in the simplified form $p^{(L)}_L \leq p^{(L)}_{L,\text{Pauli}}(p) + \Delta^{(L)}_{\text{NP}}$, as claimed. ☐

Combining Assumption 4.22, Definition 4.23, and Lemma 4.24, we obtain an explicit inequality

$$p^{(L)}_L \;\leq\; f_{\text{th}}(d_L, p) + C_{\text{loc}}\Big(\epsilon_0 + \epsilon_{\text{twirl}}\Big) + \epsilon_{\text{corr}}(n_L) + C_{\text{leak}}\varepsilon_{\text{ps}}. \tag{110}$$

Since the logical Thesis valuation is $v_{\text{Th}}(\rho_{\text{out}}; \varphi^{(L)}_L) = 1 - p^{(L)}_L$ (Definition 2.9), this gives a direct expression for v_{Th} in terms of the code distance d_L and the IQR noise parameters.

Process sector: holosymmetric valuation and δ

We next introduce an explicit dependence of the multi-cycle holosymmetric valuation $v^{(L,T)}_{\text{Holo}}$ on the differential noise parameter δ of Definition 2.18. This will be proved in detail in later phases using the Pauli-diagonal process-overlap structure induced by the IQR-to-Pauli reduction [8].

Assumption 4.25 (Holosymmetric process-overlap bound)**.** *For each code size L and time horizon T, consider two QEC schedules A, B at size L as in Definition 2.11, and their effective local Pauli noise descriptions obtained via Lemma 4.19, with per-site Pauli channels $\Lambda_{j,A}$ and $\Lambda_{j,B}$ whose single-qubit error distributions have differential parameter δ (Definition 2.18).*

Assume there exist constants $C_{\text{proc}}, \gamma_{\text{proc}} > 0$ and a sequence $\{m_L\}_L$ (e.g. the number of PQS involved in the dual-history gadget) with $m_L = O(d_L)$ such that the multi-cycle holosymmetric valuation satisfies

$$1 - v^{(L,T)}_{\text{Holo}} \;\leq\; C_{\text{proc}}\, m_L\, T\, \delta + g^{(L,T)}_{\text{NP}}, \tag{111}$$

for all $T \leq \operatorname{poly}(d_L)$, where $g^{(L,T)}_{\text{NP}}$ is a correction term arising from the same non-Pauli deviations and leakage as in the logical sector, with a bound of the form

$$g^{(L,T)}_{\text{NP}} \;\leq\; C'_{\text{proc}}\, T\Big(\epsilon_0 + \epsilon_{\text{twirl}} + \varepsilon_{\text{ps}} + \epsilon_{\text{corr}}(n_L)\Big) \tag{112}$$

for some constant $C'_{\mathrm{proc}} > 0$ *independent of* L*.*

Heuristically, Assumption 4.25 expresses that the process-level distinguishability between the A and B histories, as measured by the holosymmetric interferometer of Definition 2.12, grows at most linearly in the number of sites and cycles, with proportionality constant set by the differential noise parameter δ plus the same small non-Pauli corrections already controlled in the logical sector. This is fully consistent with process-overlap expressions of the form $v^{(L,T)}_{\mathrm{Holo}} = \frac{1}{2}(1 + C^{(L,T)})$ for suitable overlap scalars $C^{(L,T)}$ derived from Pauli-diagonal noise [1, 5, 8].

Quantitative IQR Threshold Theorem

We now assemble the logical and process sectors into a single threshold statement with explicit dependence on the IQR noise parameters and the target accuracy/runtime.

Theorem 4.26 (Target IQR fault-tolerance threshold (quantitative form))**.** *Work in the ambient IQR setting of Assumption 2.1 and the emergent Twofold register construction of Phase 1 (in particular Lemmas 3.7, 3.9, Definitions 3.10, 3.15, and Lemmas 3.18, 3.19) [8].*

Assume that the Fourfold noise kernel $\Delta\hat{G}$ *belongs to the IQR noise class of Assumption 4.5, and that the approximate local Pauli covariance of Assumption 4.13 holds. Let the IQR-to-Pauli reduction of Phase 2 apply, in the precise sense of Lemma 4.19 and Corollary 4.20, so that in one QEC cycle the emergent register is subject to an effective noise model*

$$\mathcal{N}^{(n)}_{\mathrm{eff}} = \left(\bigotimes_{j=1}^{n} \Lambda_j\right) + \mathcal{N}^{(n)}_{\mathrm{corr}} + \mathcal{N}^{(n)}_{\mathrm{leak}},$$

where each Λ_j *is a single-qubit Pauli channel with local error rate* $p^{(j)}$*,* $\mathcal{N}^{(n)}_{\mathrm{corr}}$ *is an exponentially clustered correlated tail, and* $\mathcal{N}^{(n)}_{\mathrm{leak}}$ *collects leakage into the* (p, s) *sectors.*

Assume, furthermore, that the chosen surface-code family obeys the explicit exponential logical scaling of Assumption 4.28, i.e. there exist constants

$$p_{\mathrm{th}} > 0, \qquad C_0 > 0, \qquad \gamma > 0$$

such that for local Pauli noise with error rate $p < p_{\mathrm{th}}$ *one has*

$$p^{(L)}_{L,\mathrm{Pauli}}(p) \;\le\; C_0 e^{-\gamma d_L} \qquad \textit{for all code distances } d_L. \tag{113}$$

Let $m_{\max} > 0$ *be a uniform upper bound on the number of PQS touched by a single dual-history holosymmetric gadget per QEC cycle, and let* $C_{\mathrm{loc}}, C_{\mathrm{corr}}, C_{\mathrm{leak}}, C_{\mathrm{proc}}, C'_{\mathrm{proc}} > 0$ *be the architecture-dependent constants entering:*

- *the local IQR-to-Pauli approximation (Lemma 4.15 and Corollary 4.20);*
- *the correlated tail bound* $\|\mathcal{N}^{(n)}_{\mathrm{corr}}\|_{\diamond} \le \epsilon_{\mathrm{corr}}(n)$ *with* $\epsilon_{\mathrm{corr}}(n) \le C_{\mathrm{corr}}\, n\, e^{-r/\xi}$ *(Lemma 4.18);*
- *the leakage contribution* $\|\mathcal{N}^{(n)}_{\mathrm{leak}}\|_{\diamond} \le C_{\mathrm{leak}}\, \varepsilon_{\mathrm{ps}}$ *(Phase 1 leakage lemmas);*

- *the holosymmetric process-overlap bound*

$$1 - v_{\mathrm{Holo}}^{(L,T)} \;\le\; C_{\mathrm{proc}}\, m_{\max}\, T\, \delta + C'_{\mathrm{proc}}\, T\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}} + \varepsilon_{\mathrm{ps}} + \epsilon_{\mathrm{corr}}(n_L)\Big), \tag{114}$$

  for dual-history comparisons over T cycles, as derived from the local Pauli description of the schedules A, B and the IQR noise parameters [8].

Fix:

- *a target logical accuracy $0 < \epsilon_{\log} < 1/4$,*
- *a target process accuracy $0 < \epsilon_{\mathrm{proc}} < 1/4$,*
- *and a target time horizon (number of QEC cycles) $T_{\max} \in \mathbb{N}$ for which process equivalence is required.*

Define the following derived noise thresholds*:*

$$p_* := \frac{1}{2}\, p_{\mathrm{th}}, \tag{115}$$

$$\epsilon_{\mathrm{small}} := \min\left\{ \frac{\epsilon_{\log}}{8C_{\mathrm{loc}}}, \frac{\epsilon_{\log}}{8C_{\mathrm{leak}}}, \frac{\epsilon_{\mathrm{proc}}}{4C'_{\mathrm{proc}} T_{\max}} \right\}, \tag{116}$$

$$\delta_* := \frac{\epsilon_{\mathrm{proc}}}{2C_{\mathrm{proc}}\, m_{\max}\, T_{\max}}. \tag{117}$$

Let ϵ_0, $\epsilon_{\mathrm{twirl}}$ be the smallness parameters governing the Twofold-block noise and approximate Pauli covariance in Assumptions 4.5 and 4.13, and let $\varepsilon_{\mathrm{ps}}$ be the leakage parameter of Definition 3.15. Suppose the IQR noise parameters satisfy:

(N1) (Logical physical error rate below threshold.) The effective local Pauli error rate p derived in Phase 2 obeys

$$p \;\le\; p_*.$$

(N2) (Small Twofold noise, twirling error, and leakage.) The IQR-level smallness parameters obey

$$\epsilon_0 + \epsilon_{\mathrm{twirl}} \;\le\; \epsilon_{\mathrm{small}}, \qquad \varepsilon_{\mathrm{ps}} \;\le\; \epsilon_{\mathrm{small}}. \tag{118}$$

(N3) (Small differential noise for dual histories.) The differential noise parameter satisfies

$$\delta \;\le\; \delta_*. \tag{119}$$

(N4) (Sufficient clustering for a given patch size.) For the chosen patch size n_L (and corresponding code distance d_L), the correlated tail satisfies

$$\epsilon_{\mathrm{corr}}(n_L) \;\le\; \frac{\epsilon_{\log}}{4}, \tag{120}$$

e.g. by enforcing $n_L e^{-r/\xi} \le \epsilon_{\log}/(4C_{\mathrm{corr}})$ in terms of the clustering length ξ and separation scale r.

Finally, choose the code distance d_L large enough that the ideal surface-code logical error obeys

$$C_0 e^{-\gamma d_L} \;\le\; \frac{\epsilon_{\log}}{4}, \tag{121}$$

i.e.

$$d_L \;\ge\; \frac{1}{\gamma}\log\!\Big(\frac{4C_0}{\epsilon_{\log}}\Big). \tag{122}$$

Then the following conclusions hold:

(C1) Logical sector. *The worst-case logical error probability $p_L^{(L)}$ at code size L obeys*

$$p_L^{(L)} \;\le\; C_0 e^{-\gamma d_L} + C_{\mathrm{loc}}\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}}\Big) + \epsilon_{\mathrm{corr}}(n_L) + C_{\mathrm{leak}}\,\varepsilon_{\mathrm{ps}} \;\le\; \epsilon_{\log}. \tag{123}$$

Equivalently, for the logical correctness proposition $\varphi_L^{(L)}$,

$$v_{\mathrm{Th}}\big(\rho_{\mathrm{out}};\varphi_L^{(L)}\big) \;=\; 1 - p_L^{(L)} \;\ge\; 1 - \epsilon_{\log}. \tag{124}$$

(C2) Process sector over a finite horizon. *For all $1 \le T \le T_{\max}$, the holosymmetric valuation of the process-equivalence proposition $\varphi_{\mathrm{proc}}^{(L)}$ obeys*

$$\begin{aligned} 1 - v_{\mathrm{Holo}}^{(L,T)} &\le C_{\mathrm{proc}}\, m_{\max}\, T\,\delta + C'_{\mathrm{proc}}\, T\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}} + \varepsilon_{\mathrm{ps}} + \epsilon_{\mathrm{corr}}(n_L)\Big) && (125)\\ &\le C_{\mathrm{proc}}\, m_{\max}\, T_{\max}\,\delta_* + C'_{\mathrm{proc}}\, T_{\max}\,\Big(3\epsilon_{\mathrm{small}} + \epsilon_{\log}/4\Big) && (126)\\ &\le \frac{\epsilon_{\mathrm{proc}}}{2} + \frac{\epsilon_{\mathrm{proc}}}{2} \;=\; \epsilon_{\mathrm{proc}}. && (127) \end{aligned}$$

Hence

$$v_{\mathrm{Holo}}^{(L,T)} \;\ge\; 1 - \epsilon_{\mathrm{proc}} \qquad \textit{for all } 1 \le T \le T_{\max}. \tag{128}$$

In particular, for any prescribed logical and process tolerances $(\epsilon_{\log}, \epsilon_{\mathrm{proc}})$ and any finite time horizon $T_{\max}$, there is a sufficient region *in the space of IQR noise parameters $(p, \delta, \varepsilon_{\mathrm{ps}})$, given by conditions (N1)–(N4), and a corresponding minimal code distance d_L such that the logical correctness proposition $\varphi_L^{(L)}$ and the process-equivalence proposition $\varphi_{\mathrm{proc}}^{(L)}$ both have Thesis valuations at least $1 - \epsilon_{\log}$ and $1 - \epsilon_{\mathrm{proc}}$, respectively, for all $T \le T_{\max}$.*

Remark 4.27.

- The dependence on the IQR noise parameters is now explicit:
  - p must lie below $p_* = \frac{1}{2}p_{\mathrm{th}}$, the standard Pauli threshold inherited from the surface-code analysis [1, 22, 4, 5];
  - the IQR-level smallness parameters $(\epsilon_0, \epsilon_{\mathrm{twirl}}, \varepsilon_{\mathrm{ps}})$ must all be $O(\epsilon_{\log})$ and $O(\epsilon_{\mathrm{proc}}/T_{\max})$ as enforced by $\epsilon_{\mathrm{small}}$;
  - the differential noise δ must scale at most as $O(\epsilon_{\mathrm{proc}}/T_{\max})$, with the prefactor set by the gadget size $m_{\max}$ and the process constant C_{proc}.

- For fixed physical noise parameters satisfying $p < p_*$, $\delta < \delta_*$ and $\varepsilon_{\text{ps}} < \epsilon_{\text{small}}$, the logical failure probability $p_L^{(L)}$ decays exponentially in d_L, while the process deviation $1 - v_{\text{Holo}}^{(L,T)}$ grows at most linearly in T with a slope controlled by δ and the same IQR-level smallness parameters. This matches the qualitative structure of the informal threshold Theorem 2.19, but now with explicit inequalities and constants.
- If one wishes to let $T_{\max}$ grow polynomially with d_L, the theorem exhibits the expected trade-off: for fixed ϵ_{proc} and fixed process constants $(C_{\text{proc}}, m_{\max})$, the admissible region in the $(\delta, \epsilon_0, \epsilon_{\text{twirl}}, \varepsilon_{\text{ps}})$ space shrinks in proportion to $1/T_{\max}$, exactly as one expects from standard long-time fault-tolerance considerations [2, 3, 5, 8].

In Tetralemma semantics, Theorem 4.26 thus provides, for any prescribed logical/process accuracies and finite runtime, an explicit region in IQR correlator-parameter space guaranteeing that the Thesis valuations of the logical and process propositions are both close to unity, with the AntiThesis, Synthesis, and Holothesis weights bounded in terms of $(\epsilon_{\log}, \epsilon_{\text{proc}})$.

Concrete exponential threshold scaling

To make the dependence on the code distance d_L and the IQR noise parameters fully explicit, we now specialize the abstract threshold function $f_{\text{th}}(d_L, p)$ of Assumption 4.22 to a standard exponential form, as obtained in surface-code threshold analyses for local Pauli noise [1, 22, 4, 5].

Assumption 4.28 (Concrete exponential surface-code scaling)**.** *There exist constants*

$$p_{\text{th}} > 0, \qquad C_0 > 0, \qquad \gamma > 0,$$

depending only on the chosen surface-code family and the local gadget architecture (but not on L), such that the logical error probability under local Pauli noise satisfies

$$p_{L,\text{Pauli}}^{(L)}(p) \;\le\; C_0 \exp\Big(-\gamma d_L\Big) \qquad \textit{for all } p < p_{\text{th}}. \tag{129}$$

Equivalently, one may take

$$p_{L,\text{Pauli}}^{(L)}(p) \;\le\; C_0 \left(\frac{p}{p_{\text{th}}}\right)^{\kappa d_L} \tag{130}$$

for some $\kappa > 0$, as in standard derivations where the logical error is controlled by the probability of minimum-length error chains of weight $\Omega(d_L)$ [1, 22, 4]. In particular, this gives a concrete realization of Assumption 4.22 with

$$f_{\text{th}}(d_L, p) := C_0 \exp\Big(-\gamma d_L\Big), \qquad p < p_{\text{th}}. \tag{131}$$

We will henceforth take $p_* < p_{\text{th}}$ in Theorem 2.19 and work in the regime $p \le p_*$ so that

$$p_{L,\text{Pauli}}^{(L)}(p) \;\le\; C_0 \exp(-\gamma d_L)$$

holds uniformly over all L.

Explicit IQR threshold with exponential decay

We now specialize Theorem 2.19 to the concrete scaling of Assumption 4.28. This yields an explicit choice of code distance d_L and constraints on $(p, \delta, \varepsilon_{\mathrm{ps}})$ in terms of a target accuracy ϵ.

Corollary 4.29 (Explicit IQR threshold with exponential logical decay)**.** *Assume the hypotheses of Theorem 2.19 and, in addition, the concrete scaling of Assumption 4.28. Fix a target accuracy* $\epsilon > 0$*.*

(1) Logical sector. *Assume:*

(L1) The IQR-derived local Pauli error rate satisfies $p \le p_* < p_{\mathrm{th}}$ *for some* p_* *strictly below the Pauli threshold.*

(L2) The non-Pauli corrections are bounded by

$$C_{\mathrm{loc}}\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}}\Big) + C_{\mathrm{leak}}\varepsilon_{\mathrm{ps}} \;\le\; \frac{\epsilon}{4}, \tag{132}$$

and the correlated tail satisfies

$$\epsilon_{\mathrm{corr}}(n_L) \;\le\; \frac{\epsilon}{4} \tag{133}$$

for the chosen patch size n_L*.*

Then, for any L *such that*

$$d_L \;\ge\; \frac{1}{\gamma}\log\Big(\frac{4C_0}{\epsilon}\Big), \tag{134}$$

the logical error probability at size L *satisfies*

$$p_L^{(L)} \;\le\; C_0 e^{-\gamma d_L} + C_{\mathrm{loc}}\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}}\Big) + \epsilon_{\mathrm{corr}}(n_L) + C_{\mathrm{leak}}\varepsilon_{\mathrm{ps}} \;\le\; \epsilon. \tag{135}$$

Equivalently, the Thesis valuation of the logical correctness proposition $\varphi_L^{(L)}$ *obeys*

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) = 1 - p_L^{(L)} \;\ge\; 1 - \epsilon.$$

(2) Process sector. *Assume:*

(P1) The dual-history gadgets use at most m_L *PQS with* $m_L \le c_m d_L$ *for some constant* $c_m > 0$ *(i.e. the number of sites touched per cycle scales at most linearly in the distance).*

(P2) The number of QEC cycles of interest is bounded by a fixed polynomial in the distance, $T \le d_L^{\kappa_T}$ *for some* $\kappa_T > 0$*, as in standard long-time fault-tolerance formulations [2, 3, 4].*

(P3) The differential noise parameter δ *satisfies*

$$\delta \;\leq\; \frac{\epsilon}{4C_{\mathrm{proc}} c_m d_L^{\kappa_T+1}}, \tag{136}$$

and the non-Pauli process corrections obey

$$C'_{\mathrm{proc}}\, T\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}} + \varepsilon_{\mathrm{ps}} + \epsilon_{\mathrm{corr}}(n_L)\Big) \;\leq\; \frac{\epsilon}{2} \tag{137}$$

for all $T \leq d_L^{\kappa_T}$*.*

Then, for all $T \leq d_L^{\kappa_T}$*, the holosymmetric valuation satisfies*

$$1 - v_{\mathrm{Holo}}^{(L,T)} \;\leq\; C_{\mathrm{proc}}\, m_L\, T\, \delta + C'_{\mathrm{proc}}\, T\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}} + \varepsilon_{\mathrm{ps}} + \epsilon_{\mathrm{corr}}(n_L)\Big) \;\leq\; \epsilon, \tag{138}$$

so that

$$v_{\mathrm{Holo}}^{(L,T)} \;\geq\; 1 - \epsilon \qquad \textit{for all } T \leq d_L^{\kappa_T}.$$

In particular, under the combined conditions $p \leq p_* < p_{\mathrm{th}}$*,* $\delta \leq \delta_*(L, \epsilon)$ *as in (P3), and* $\varepsilon_{\mathrm{ps}} \leq \varepsilon_{\mathrm{ps},*}(\epsilon)$ *as in (L2), the logical Thesis valuation and the process-level holosymmetric valuation can both be made arbitrarily close to* 1 *by choosing the code distance* d_L *sufficiently large.*

Remark 4.30. Corollary 4.29 makes explicit:

- the exponential suppression of the logical failure probability $p_L^{(L)}$ in the code distance d_L, with rate γ inherited from the underlying surface-code threshold (Assumption 4.28);
- the polynomial constraint on the allowed time horizon $T \leq d_L^{\kappa_T}$ in which the holosymmetric Thesis valuation $v_{\mathrm{Holo}}^{(L,T)}$ remains above $1-\epsilon$, controlled by the differential noise parameter δ and the same IQR-level smallness parameters $(\epsilon_0, \epsilon_{\mathrm{twirl}}, \varepsilon_{\mathrm{ps}})$ and clustering scale ξ that enter the logical sector;
- the explicit trade-offs: for fixed ϵ one can either tighten the IQR-level physical noise parameters $(p, \delta, \varepsilon_{\mathrm{ps}})$ or increase the code distance d_L (or both) to achieve the desired valuations, in direct analogy with standard surface-code threshold analyses [1, 22, 4, 5, 8].

In Tetralemma terms, this provides a concrete regime of IQR-level physical parameters for which both the state-level proposition $\varphi_L^{(L)}$ and the process-level proposition $\varphi_{\mathrm{proc}}^{(L)}$ acquire Thesis valuations arbitrarily close to 1, with the AntiThesis, Synthesis, and Holothesis weights suppressed by explicit functions of d_L and the IQR noise parameters.

Corollary 4.31 (Simplified threshold region in $(p, \delta, \varepsilon_{\mathrm{ps}})$-space)**.** *Assume the hypotheses of Theorem 4.26, including:*

- *the ambient IQR setting and emergent register construction of Phase 1 [8];*
- *the IQR noise class conditions of Assumption 4.5;*

- *approximate local Pauli covariance as in Assumption 4.13;*
- *the IQR-to-Pauli reduction of Lemma 4.19 and Corollary 4.20;*
- *the explicit surface-code logical scaling of Assumption 4.28 with constants* $(p_{\text{th}}, C_0, \gamma)$.

Assume in addition that the IQR smallness parameters

$$\epsilon_0, \quad \epsilon_{\text{twirl}}, \quad \epsilon_{\text{corr}}(n_L)$$

appearing in Phase 2 depend continuously on the effective local Pauli error rate p *and satisfy*

$$\epsilon_0(p) \to 0, \quad \epsilon_{\text{twirl}}(p) \to 0, \quad \epsilon_{\text{corr}}(n_L; p) \to 0 \qquad \text{as } p \to 0, \tag{139}$$

uniformly in the code size L *in the regime of interest, as expected in the weak-noise limit of the IQR/Tetralemma construction [8].*

Let $m_{\max}$, C_{loc}, C_{corr}, C_{leak}, C_{proc}, C'_{proc} *be the architecture-dependent constants entering Theorem 4.26.*

Then, for every choice of

$$0 < \epsilon < \tfrac{1}{4}, \qquad T_{\max} \in \mathbb{N},$$

there exist positive thresholds

$$p_*(\epsilon, T_{\max}) > 0, \qquad \delta_*(\epsilon, T_{\max}) > 0, \qquad \varepsilon_{\text{ps},*}(\epsilon, T_{\max}) > 0, \tag{140}$$

with

$$p_*(\epsilon, T_{\max}) \leq \tfrac{1}{2} p_{\text{th}}, \tag{141}$$

such that the following holds.

If the effective IQR noise parameters satisfy

$$p \ < \ p_*(\epsilon, T_{\max}), \qquad \delta \ < \ \delta_*(\epsilon, T_{\max}), \qquad \varepsilon_{\text{ps}} \ < \ \varepsilon_{\text{ps},*}(\epsilon, T_{\max}), \tag{142}$$

then there exists a surface-code distance d_L *(and corresponding patch size* L*) for which:*

(a) Logical correctness. *The worst-case logical error probability* $p_L^{(L)}$ *after one QEC cycle obeys*

$$p_L^{(L)} \ \leq \ \epsilon, \tag{143}$$

i.e. for the logical correctness proposition $\varphi_L^{(L)}$ *one has*

$$v_{\text{Th}}(\rho_{\text{out}}; \varphi_L^{(L)}) \ = \ 1 - p_L^{(L)} \ \geq \ 1 - \epsilon. \tag{144}$$

(b) Process equivalence up to $T_{\max}$ cycles. *For all* $1 \leq T \leq T_{\max}$, *the holosymmetric valuation of the process-equivalence proposition* $\varphi_{\text{proc}}^{(L)}$ *satisfies*

$$v_{\text{Holo}}^{(L,T)} \ \geq \ 1 - \epsilon. \tag{145}$$

Equivalently, for any desired tolerance ϵ *and time horizon* $T_{\max}$*, there is a* sufficient *region*

$$\mathcal{R}_{\epsilon,T_{\max}} := \Big\{(p,\delta,\varepsilon_{\mathrm{ps}}) \in [0,1]^3 : p < p_*(\epsilon,T_{\max}),\ \delta < \delta_*(\epsilon,T_{\max}),\ \varepsilon_{\mathrm{ps}} < \varepsilon_{\mathrm{ps},*}(\epsilon,T_{\max})\Big\} \quad (146)$$

in the IQR noise-parameter space such that, once $(p,\delta,\varepsilon_{\mathrm{ps}}) \in \mathcal{R}_{\epsilon,T_{\max}}$*, one can choose a code distance* d_L *large enough for both:*

(i) state-level logical correctness and (ii) process-level holosymmetry

to hold with Tetralemma Thesis valuations at least $1-\epsilon$ *for all* $T \le T_{\max}$*.*

Proof. Starting from Theorem 4.26 with $\epsilon_{\log} = \epsilon_{\mathrm{proc}} = \epsilon$, the logical sector requires

$$C_0 e^{-\gamma d_L} + C_{\mathrm{loc}}\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}}\Big) + \epsilon_{\mathrm{corr}}(n_L) + C_{\mathrm{leak}}\,\varepsilon_{\mathrm{ps}} \;\le\; \epsilon, \quad (147)$$

while the process sector for $T \le T_{\max}$ requires

$$C_{\mathrm{proc}}\, m_{\max}\, T_{\max}\,\delta + C'_{\mathrm{proc}}\, T_{\max}\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}} + \varepsilon_{\mathrm{ps}} + \epsilon_{\mathrm{corr}}(n_L)\Big) \;\le\; \epsilon. \quad (148)$$

By continuity of $\epsilon_0(p)$, $\epsilon_{\mathrm{twirl}}(p)$, $\epsilon_{\mathrm{corr}}(n_L;p)$ at $p=0$ and the fact that they vanish as $p \to 0$, there exists $p_*(\epsilon,T_{\max}) > 0$ with $p_*(\epsilon,T_{\max}) \le \frac{1}{2}p_{\mathrm{th}}$ such that, whenever $p < p_*(\epsilon,T_{\max})$, one has

$$C_{\mathrm{loc}}\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}}\Big) + \epsilon_{\mathrm{corr}}(n_L) \;\le\; \frac{\epsilon}{4}, \qquad C'_{\mathrm{proc}}\, T_{\max}\Big(\epsilon_0 + \epsilon_{\mathrm{twirl}} + \epsilon_{\mathrm{corr}}(n_L)\Big) \;\le\; \frac{\epsilon}{4}. \quad (149)$$

Similarly, choose

$$\varepsilon_{\mathrm{ps},*}(\epsilon,T_{\max}) := \min\Big\{\frac{\epsilon}{4C_{\mathrm{leak}}}, \frac{\epsilon}{4C'_{\mathrm{proc}}T_{\max}}\Big\}, \qquad \delta_*(\epsilon,T_{\max}) := \frac{\epsilon}{2C_{\mathrm{proc}}m_{\max}T_{\max}}. \quad (150)$$

Then, if $\varepsilon_{\mathrm{ps}} < \varepsilon_{\mathrm{ps},*}(\epsilon,T_{\max})$ and $\delta < \delta_*(\epsilon,T_{\max})$, the leakage and differential-noise contributions are bounded by $\epsilon/4$ and $\epsilon/2$, respectively.

With these choices, the two inequalities above reduce to the requirement

$$C_0 e^{-\gamma d_L} \;\le\; \frac{\epsilon}{4}, \quad (151)$$

which is ensured by taking

$$d_L \;\ge\; \frac{1}{\gamma}\log\Big(\frac{4C_0}{\epsilon}\Big). \quad (152)$$

Substituting these bounds into Theorem 4.26 gives $p_L^{(L)} \le \epsilon$ and $1 - v_{\mathrm{Holo}}^{(L,T)} \le \epsilon$ for all $T \le T_{\max}$, i.e. the claimed logical and process valuations.

The thresholds $p_*(\epsilon,T_{\max})$, $\delta_*(\epsilon,T_{\max})$, $\varepsilon_{\mathrm{ps},*}(\epsilon,T_{\max})$ are strictly positive by continuity and depend only on the target $(\epsilon,T_{\max})$ and on the fixed architectural constants, not on the particular code instance beyond its distance d_L. This defines the region $\mathcal{R}_{\epsilon,T_{\max}}$ and completes the proof. □

Corollary 4.32 (Distance-dependent differential-noise threshold)**.** *Assume the hypotheses of Theorem 4.26, including:*

- *the ambient IQR setting and emergent register construction of Phase 1 [8];*
- *the IQR noise-class conditions of Assumption 4.5;*
- *approximate local Pauli covariance as in Assumption 4.13;*
- *the IQR-to-Pauli reduction and leakage control of Corollary 4.20;*
- *explicit surface-code scaling as in Assumption 4.28 with constants* $(p_{\mathrm{th}}, C_0, \gamma)$.

Let $m_{\max}$, C_{loc}, C_{corr}, C_{leak}, C_{proc}, C'_{proc} *be the architecture-dependent constants entering Theorem 4.26.*

Fix a target accuracy $0 < \epsilon < \frac{1}{4}$ *and let the maximal number of QEC cycles be parameterized as a polynomial in the code distance,*

$$T_{\max}(d_L) := c_T\, d_L^{\kappa}, \tag{153}$$

for some fixed constants $c_T \geq 1$ *and* $\kappa \geq 1$ *independent of* L*, as in standard long-time threshold settings [2, 3, 4, 5].*

Assume that the effective local Pauli error rate p *and the leakage parameter* $\varepsilon_{\mathrm{ps}}$ *satisfy the bounds*

$$p < p_*(\epsilon), \qquad \varepsilon_{\mathrm{ps}} < \varepsilon_{\mathrm{ps},*}(\epsilon), \tag{154}$$

for some thresholds $p_*(\epsilon) > 0$, $\varepsilon_{\mathrm{ps},*}(\epsilon) > 0$ *chosen so that, for all distances* d_L *in the regime of interest,*

$$C_{\mathrm{loc}}\Big(\epsilon_0(p) + \epsilon_{\mathrm{twirl}}(p)\Big) + \epsilon_{\mathrm{corr}}(n_L; p) \;\leq\; \frac{\epsilon}{4}, \tag{155}$$

$$C_{\mathrm{leak}}\, \varepsilon_{\mathrm{ps}} \;\leq\; \frac{\epsilon}{4}, \tag{156}$$

and similarly

$$C'_{\mathrm{proc}}\, T_{\max}(d_L)\Big(\epsilon_0(p) + \epsilon_{\mathrm{twirl}}(p) + \epsilon_{\mathrm{corr}}(n_L; p) + \varepsilon_{\mathrm{ps}}\Big) \;\leq\; \frac{\epsilon}{4} \tag{157}$$

for all d_L *considered. (This is possible because* $\epsilon_0(p)$, $\epsilon_{\mathrm{twirl}}(p)$, $\epsilon_{\mathrm{corr}}(n_L; p)$ *all vanish as* $p \to 0$*, uniformly in* L*, in the weak-noise IQR limit [8].)*

Define the distance-dependent differential-noise threshold

$$\delta_*(d_L; \epsilon) := \frac{\epsilon}{4\, C_{\mathrm{proc}}\, m_{\max}\, T_{\max}(d_L)} \;=\; \frac{\epsilon}{4\, C_{\mathrm{proc}}\, m_{\max}\, c_T}\, d_L^{-\kappa}. \tag{158}$$

Then, for any code distance d_L *and any effective differential noise parameter* δ *satisfying*

$$\delta \;<\; \delta_*(d_L; \epsilon), \tag{159}$$

the following hold simultaneously:

(a) Logical correctness. *The worst-case logical error probability after one QEC cycle obeys*

$$p_L^{(L)} \;\leq\; \epsilon, \tag{160}$$

hence the Tetralemma Thesis valuation of the logical correctness proposition $\varphi_L^{(L)}$ *satisfies*

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) \;=\; 1 - p_L^{(L)} \;\geq\; 1 - \epsilon. \tag{161}$$

(b) Process holosymmetry up to $T_{\max}(d_L)$ cycles. *For all integers* $1 \leq T \leq T_{\max}(d_L)$*, the holosymmetric Thesis valuation of the process-equivalence proposition* $\varphi^{(L)}_{\mathrm{proc}}$ *satisfies*

$$v^{(L,T)}_{\mathrm{Holo}} \;\geq\; 1 - \epsilon. \tag{162}$$

In particular, along any sequence of codes with distance $d_L \to \infty$ *and* $T_{\max}(d_L) = c_T d_L^{\kappa}$ *fixed as above, the differential noise parameter can be allowed to scale as*

$$\delta = O\Big(d_L^{-\kappa}\Big) \tag{163}$$

and still guarantees both logical correctness and process-level holosymmetry with Tetralemma Thesis weight at least $1 - \epsilon$*.*

Proof. The logical sector estimate in Theorem 4.26 reads

$$p^{(L)}_L \;\leq\; C_0 e^{-\gamma d_L} + C_{\mathrm{loc}}\Big(\epsilon_0(p) + \epsilon_{\mathrm{twirl}}(p)\Big) + \epsilon_{\mathrm{corr}}(n_L; p) + C_{\mathrm{leak}}\, \varepsilon_{\mathrm{ps}}. \tag{164}$$

By (155) and (156), the last three terms in (164) contribute at most $\epsilon/4 + \epsilon/4 = \epsilon/2$. Choosing

$$d_L \;\geq\; \frac{1}{\gamma} \log\Big(\frac{2C_0}{\epsilon}\Big) \tag{165}$$

ensures the first term is at most $\epsilon/2$, and hence $p^{(L)}_L \leq \epsilon$.

For the process sector, the quantitative bound of Theorem 4.26 gives, for each $1 \leq T \leq T_{\max}(d_L)$,

$$1 - v^{(L,T)}_{\mathrm{Holo}} \;\leq\; C_{\mathrm{proc}}\, m_{\max}\, T\, \delta + C'_{\mathrm{proc}}\, T\Big(\epsilon_0(p) + \epsilon_{\mathrm{twirl}}(p) + \epsilon_{\mathrm{corr}}(n_L; p) + \varepsilon_{\mathrm{ps}}\Big). \tag{166}$$

Since $T \leq T_{\max}(d_L)$ and the bracketed expression is bounded as in (157), the second term on the right-hand side of (166) is at most $\epsilon/4$. If we now impose

$$C_{\mathrm{proc}}\, m_{\max}\, T_{\max}(d_L)\, \delta \;\leq\; \frac{\epsilon}{4}, \tag{167}$$

then in particular $C_{\mathrm{proc}}\, m_{\max}\, T\, \delta \leq \epsilon/4$ for all $T \leq T_{\max}(d_L)$. This is exactly the condition

$$\delta \;\leq\; \frac{\epsilon}{4 C_{\mathrm{proc}} m_{\max} T_{\max}(d_L)} \;=\; \delta_*(d_L; \epsilon),$$

by definition of $\delta_*(d_L; \epsilon)$.

Under this choice,

$$1 - v^{(L,T)}_{\mathrm{Holo}} \;\leq\; \frac{\epsilon}{4} + \frac{\epsilon}{4} \;=\; \frac{\epsilon}{2} \;<\; \epsilon \tag{168}$$

for all $1 \leq T \leq T_{\max}(d_L)$, which implies $v^{(L,T)}_{\mathrm{Holo}} \geq 1 - \epsilon$ and establishes the process claim. The scaling $\delta_*(d_L; \epsilon) \propto d_L^{-\kappa}$ is immediate from the definition of $T_{\max}(d_L)$, completing the proof. □

5 Phase 3: Code Family, Stabilizer Robustness, and Logical Threshold

In this section we place a family of surface codes on the emergent Twofold (qubit) register constructed in Phase 1, define a stabilizer-based robustness functional for these codes under the effective local Pauli noise obtained in Phase 2, and recall / adapt logical threshold estimates for the surface-code family, in line with the standard topological-stabilizer and surface-code literature [16, 21, 1, 22, 4, 15, 5].

Throughout, we work under the assumptions and constructions of Sections 3–4 within the IQR/Holo-State framework [8]. In particular:

- modal fibres $\mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)} \cong \mathbb{C}^4$ and orthonormal Fourfold bases $\{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$ are induced from the coincident-limit inner products of Chapter 4 [8] via Assumptions 3.3 and 3.6;
- the Twofold subspaces $\mathcal{H}_{(0,1),x} \subset \mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)}$ form a smooth rank-2 sub-bundle over $\mathcal{U}$ (Lemma 3.9);
- the emergent register and its leakage parameter $\varepsilon_{\mathrm{ps}}$ are as in Phase 1 (Definitions 3.10 and 3.15);
- the effective local Pauli noise acting on the Twofold fibres is as in Corollary 4.20, obtained from the Fourfold correlator via the IQR-to-Pauli reduction of Phase 2.

5.1 Surface-Code Family on the Emergent Twofold Lattice

We first specify a family of surface-code patches $\{\mathcal{C}_L\}_{L\in\mathbb{N}}$ on the emergent Twofold register, modelled on the usual rotated planar surface-code construction in a 2D lattice geometry with local stabilizer checks [1, 22, 4, 15].

Definition 5.1 (Rotated lattice patches in the emergent geometry)**.** For each integer $L \geq 1$, choose a finite set of PQS loci

$$\mathcal{P}_L = \{x_j^{(L)}\}_{j=1}^{n_L} \subset \mathcal{U} \subset \mathcal{X}_q, \tag{169}$$

such that:

- the loci $x_j^{(L)}$ lie in a region of the PQS continuum where the modal Gram matrices are non-degenerate and smoothly varying, as in Assumption 3.6 and Chapter 4 of [8];
- with respect to the correlation-distance pseudometric d_{corr} of Assumption 4.3, the adjacency graph of $\mathcal{P}_L$ is (up to bounded geometric distortions) a rotated $L \times L$ square lattice with appropriate boundary conditions (e.g. “rough” and “smooth” boundaries on opposite sides) [1, 4].

We denote by $n_L := |\mathcal{P}_L|$ the number of PQS in the patch. For a standard rotated planar surface code one has

$$n_L = \Theta(L^2). \tag{170}$$

Definition 5.2 (Emergent register at size L)**.** For each L, the emergent Twofold register on $\mathcal{P}_L$ is the tensor product

$$\mathcal{H}_{\mathrm{reg}}^{(L)} := \bigotimes_{x_j \in \mathcal{P}_L} \mathcal{H}_{(0,1),x_j} \;\cong\; (\mathbb{C}^2)^{\otimes n_L}, \tag{171}$$

in direct continuation of the emergent-register construction of Definition 3.10 and Section 3 [8]. The physical qubits of the surface code are thus the emergent Twofold modes at the chosen PQS loci.

Definition 5.3 (Surface code at size L)**.** A *surface code* at size L on the emergent register $\mathcal{H}_{\mathrm{reg}}^{(L)}$ consists of:

- a code subspace $\mathcal{C}_L \subset \mathcal{H}_{\mathrm{reg}}^{(L)}$ encoding one logical qubit:

  $$\dim \mathcal{C}_L = 2;$$

- a stabilizer group $S_L \subset \mathcal{PU}\Big(\mathcal{H}_{\mathrm{reg}}^{(L)}\Big)$, which is an abelian subgroup of the n_L-qubit Pauli group (excluding $-I$), generated by a set of local "star" (X-type) and "plaquette" (Z-type) check operators of bounded weight (typically weight 4 away from boundaries) supported on small clusters of PQS in the rotated-lattice geometry of Definition 5.1, as in the standard surface-code constructions [16, 21, 1, 4, 15];

- a code projector

  $$P_{\mathrm{SC},L} := \frac{1}{|S_L|} \sum_{g \in S_L} g, \tag{172}$$

  which projects onto the joint $+1$ eigenspace of all stabilizers and coincides with the logical code space: $\mathcal{C}_L = P_{\mathrm{SC},L}\, \mathcal{H}_{\mathrm{reg}}^{(L)}$.

We denote by d_L the *code distance* at size L, i.e. the minimal weight of any Pauli operator on $\mathcal{H}_{\mathrm{reg}}^{(L)}$ that commutes with all stabilizers but is not itself in S_L (a nontrivial logical operator). For standard rotated planar surface-code families one has

$$d_L \sim L, \tag{173}$$

reflecting the topological protection provided by extended logical string operators across the emergent Twofold lattice [1, 4, 5].

Remark 5.4. The family $\{\mathcal{C}_L\}$ defined above is a *topological stabilizer-code family* in the usual sense: local checks of constant weight on a 2D lattice, with asymptotically growing distance d_L and constant encoding rate (one logical qubit for $n_L = \Theta(L^2)$ physical qubits), as in the toric/surface-code and related topological constructions [16, 21, 1, 15, 5]. Crucially, in the present IQR setting the underlying 2D lattice is realized as a patch of PQS loci in the emergent geometry of the Holo-State [8], and the stabilizer generators act as dressed Pauli operators on the Twofold fibres $\mathcal{H}_{(0,1),x_j}$, consistent with the local Pauli noise model derived in Phase 2.

5.2 IQR-Derived Noise Parameters and Stabilizer Robustness

We now make explicit how the IQR noise parameters $(p, \delta, \varepsilon_{\mathrm{ps}})$ of Definition 2.18 and the IQR-to-Pauli reduction of Phase 2 feed into the effective *physical* noise seen by the surface-code family $\{\mathcal{C}_L\}_{L\in\mathbb{N}}$ on the emergent Twofold register. This will allow us to connect directly to standard logical-threshold estimates for surface codes under local Pauli noise [1, 22, 4, 5].

Throughout this subsection we fix a code size L and the corresponding patch $\mathcal{P}_L$, emergent register $\mathcal{H}^{(L)}_{\mathrm{reg}}$, and surface code $\mathcal{C}_L$ as in Definitions 5.1–5.3.

Definition 5.5 (Inherited local noise parameters at size L)**.** Let $\mathcal{N}^{(n_L)}$ be the global register channel for one QEC cycle on $\mathcal{H}^{(L)}_{\mathrm{reg}}$ (with $n_L = |\mathcal{P}_L|$), and let

$$\mathcal{N}^{(n_L)}_{\mathrm{eff}} = \left(\bigotimes_{j=1}^{n_L} \Lambda_j\right) + \mathcal{N}^{(n_L)}_{\mathrm{corr}} + \mathcal{N}^{(n_L)}_{\mathrm{leak}}$$

be the effective noise decomposition from Corollary 4.20, where:

- Λ_j is the single-qubit Pauli channel on $\mathcal{H}_{(0,1),x_j}$ with error rate $p^{(j)} := 1 - q_0^{(j)}$;
- $\mathcal{N}^{(n_L)}_{\mathrm{corr}}$ is the correlated tail with diamond norm $\|\mathcal{N}^{(n_L)}_{\mathrm{corr}}\|_\diamond \le \epsilon_{\mathrm{corr}}(n_L)$ and $\epsilon_{\mathrm{corr}}(n_L) = O\Big(n_L e^{-r/\xi}\Big)$ as in Lemma 4.18;
- $\mathcal{N}^{(n_L)}_{\mathrm{leak}}$ collects contributions from leakage into (p, s) sectors, with $\|\mathcal{N}^{(n_L)}_{\mathrm{leak}}\|_\diamond = O(\varepsilon_{\mathrm{ps}})$ as in Corollary 4.20.

We define the *inherited local noise parameters* for the surface-code patch at size L by

$$p^{(L)}_{\mathrm{loc}} := \max_{1\le j\le n_L} p^{(j)}, \tag{174}$$

$$\epsilon^{(L)}_{\mathrm{corr}} := \|\mathcal{N}^{(n_L)}_{\mathrm{corr}}\|_\diamond, \tag{175}$$

$$\epsilon^{(L)}_{\mathrm{leak}} := \|\mathcal{N}^{(n_L)}_{\mathrm{leak}}\|_\diamond. \tag{176}$$

By Corollary 4.20 and Lemma 4.18, these satisfy

$$p^{(L)}_{\mathrm{loc}} \le p, \qquad \epsilon^{(L)}_{\mathrm{corr}} = O\Big(n_L e^{-r/\xi}\Big), \qquad \epsilon^{(L)}_{\mathrm{leak}} = O(\varepsilon_{\mathrm{ps}}). \tag{177}$$

Definition 5.6 (Effective physical error rate on the patch)**.** For each code size L we define the *effective physical error rate* seen by the surface code on $\mathcal{P}_L$ to be

$$p^{(L)}_{\mathrm{eff}} := p^{(L)}_{\mathrm{loc}} + \alpha_{\mathrm{corr}}\, \epsilon^{(L)}_{\mathrm{corr}} + \alpha_{\mathrm{leak}}\, \epsilon^{(L)}_{\mathrm{leak}}, \tag{178}$$

where $\alpha_{\mathrm{corr}}, \alpha_{\mathrm{leak}} > 0$ are fixed constants depending only on the local gadget architecture and the dimension of $\mathcal{H}^{(L)}_{\mathrm{reg}}$ (via standard norm equivalences between channel diamond norms and worst-case failure probabilities; cf. Lemma 3.18 and Corollary 4.20). Equivalently,

$$p^{(L)}_{\mathrm{eff}} = p^{(L)}_{\mathrm{loc}} + O\Big(n_L e^{-r/\xi}\Big) + O(\varepsilon_{\mathrm{ps}}). \tag{179}$$

Remark 5.7. By Assumption 4.5 and the construction of the local Pauli channels Λ_j in Lemmas 4.11–4.15, the basic physical error rate p of Definition 2.18 controls the deviations of the Twofold block from its ideal evolution, while the leakage parameter ε_{ps} controls mixing into (p, s) sectors, and the clustering data (ξ, r) control the correlated tail. Thus $p_{\text{eff}}^{(L)}$ is an explicit IQR-derived upper bound on the per-qubit physical error rate seen by the surface code at size L, including corrections from correlations and leakage, directly inherited from the correlator-level IQR noise parameters $(p, \delta, \varepsilon_{\text{ps}})$ [8]. The parameter δ will enter explicitly in the process / holosymmetry sector (Phase 4) rather than the state-level logical error analysis.

We now package the standard surface-code logical threshold behaviour in terms of the effective rate $p_{\text{eff}}^{(L)}$.

Definition 5.8 (Stabilizer robustness functional)**.** Let $p_{\text{eff}}^{(L)}$ be as in Definition 5.6. The *stabilizer robustness functional* for the surface code $\mathcal{C}_L$ is the map

$$\mathcal{R}_L : [0,1] \to [0,1], \qquad \mathcal{R}_L(p) := \sup\Big\{ p_L^{(L)}(\mathcal{N}) \;\Big|\; \mathcal{N} \text{ a CPTP map on } \mathcal{H}_{\text{reg}}^{(L)} \Big\}, \tag{180}$$
$$\text{with } p_{\text{eff}}^{(L)}(\mathcal{N}) \le p$$

where $p_L^{(L)}(\mathcal{N})$ is the worst-case logical error probability for one QEC cycle under $\mathcal{N}$ as in Definition 2.7, and $p_{\text{eff}}^{(L)}(\mathcal{N})$ denotes the effective physical rate for $\mathcal{N}$ computed via Definition 5.6. In words, $\mathcal{R}_L(p)$ is the worst possible logical failure probability for the size-L surface code among all noise channels whose effective physical rate does not exceed p.

Lemma 5.9 (Surface-code robustness under local Pauli noise)**.** *There exists a constant $p_* > 0$ (the surface-code threshold) and constants $c_1, c_2 > 0$, independent of L, such that for all $p < p_*$ and all sufficiently large L,*

$$\mathcal{R}_L(p) \;\le\; c_1 \exp\Big(-c_2 d_L\Big), \tag{181}$$

where d_L is the distance of $\mathcal{C}_L$ and $d_L \sim L$ as in Definition 5.3.

Proof. This is the standard surface-code threshold behaviour under local Pauli noise, as established in [1, 22, 4, 5]. For noise models of the form $\bigotimes_{j=1}^{n_L} \Lambda_j$ with per-qubit error rates $p^{(j)} \le p$, these works show that for $p < p_*$ the logical failure probability per QEC cycle decays exponentially in the distance, with $p_L^{(L)} \le c_1 \exp(-c_2 d_L)$ for suitable constants $c_1, c_2 > 0$ independent of L.

The definition of $\mathcal{R}_L(p)$ takes the supremum over all channels whose effective physical rate is at most p; by construction of $p_{\text{eff}}^{(L)}$, such channels differ from a local Pauli tensor product by a correlated tail of diamond norm at most $O\Big(n_L e^{-r/\xi}\Big)$ and a leakage contribution of order $O(\varepsilon_{\text{ps}})$. For sufficiently small p, and in the regime where $n_L e^{-r/\xi}$ and ε_{ps} are bounded by fixed multiples of p (as in the IQR noise-class assumptions and Corollary 4.20), these corrections can be absorbed into a renormalization of p below the threshold p_*, leaving the exponential decay in d_L unchanged up to a redefinition of c_1, c_2. Thus the bound holds for all channels in the admissible set, and hence for $\mathcal{R}_L(p)$. ☐

Corollary 5.10 (Inherited logical threshold condition)**.** *Let the IQR noise parameters* $(p, \delta, \varepsilon_{\mathrm{ps}})$ *satisfy*

$$p_{\mathrm{eff}}^{(L)} \;\leq\; p_{\mathrm{loc}}^{(L)} + \alpha_{\mathrm{corr}}\, \epsilon_{\mathrm{corr}}^{(L)} + \alpha_{\mathrm{leak}}\, \epsilon_{\mathrm{leak}}^{(L)} \;<\; p_* \tag{182}$$

for all sufficiently large L*, with* $p_{\mathrm{loc}}^{(L)} \leq p$ *and* $\epsilon_{\mathrm{corr}}^{(L)}, \epsilon_{\mathrm{leak}}^{(L)}$ *as in Definition 5.5. Then for any target logical accuracy* $\epsilon > 0$ *there exists* L_0 *such that for all* $L \geq L_0$*,*

$$p_L^{(L)} \;\leq\; \mathcal{R}_L\Big(p_{\mathrm{eff}}^{(L)}\Big) \;<\; \epsilon. \tag{183}$$

Equivalently, the Thesis valuation of the logical correctness proposition $\varphi_L^{(L)}$ *satisfies*

$$\upsilon_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) = 1 - p_L^{(L)} \;>\; 1 - \epsilon \tag{184}$$

for all sufficiently large L*.*

Proof. Fix $\epsilon > 0$. By Lemma 5.9, $\mathcal{R}_L(p)$ decays exponentially in $d_L \sim L$ whenever $p < p_*$. By assumption $p_{\mathrm{eff}}^{(L)} < p_*$ for all sufficiently large L, so there exists L_0 such that for all $L \geq L_0$,

$$\mathcal{R}_L\Big(p_{\mathrm{eff}}^{(L)}\Big) \leq c_1 \exp(-c_2 d_L) < \epsilon.$$

By Definition 5.8, $p_L^{(L)} \leq \mathcal{R}_L(p_{\mathrm{eff}}^{(L)})$ for the actual noise channel on the emergent register, yielding the claimed bound on $p_L^{(L)}$ and hence on the Thesis valuation $\upsilon_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)})$ via Definition 2.9. □

5.3 Stabilizer Robustness Polynomial

We now define a stabilizer-based robustness functional for the surface-code family under local Pauli noise, and relate it to combinatorial weight data of the stabilizer group, in the spirit of weight-enumerator- and syndrome-based treatments of stabilizer codes [19, 13, 5].

In this subsection we work with the emergent Twofold register $\mathcal{H}_{\mathrm{reg}}^{(L)}$ constructed in Definition 5.2, itself arising from the Twofold sub-bundle of the modal fibre bundle over a PQS patch as in Phase 1 and the corresponding constructions in [8, Chs. 4,9].

Local Pauli noise and shrink factors

Definition 5.11 (Single-qubit Pauli channel with shrink factors)**.** A single-qubit Pauli channel Λ on $\mathcal{B}(\mathbb{C}^2)$ is a CPTP map of the form

$$\Lambda(\rho) := \sum_{P \in \{\mathbb{I}, X, Y, Z\}} q_P\, P \rho P, \qquad q_P \geq 0,\ \sum_P q_P = 1, \tag{185}$$

as in standard Pauli-noise models for fault-tolerant QEC [13, 1, 5]. Its action on the Pauli basis is diagonal:

$$\Lambda(P) = \lambda_P P, \quad P \in \{X, Y, Z\}, \tag{186}$$

with *shrink factors*

$$\lambda_P = q_{\mathbb{I}} + q_P - \sum_{P' \neq P} q_{P'} = 2\Big(q_{\mathbb{I}} + q_P\Big) - 1, \qquad P \in \{X, Y, Z\}, \tag{187}$$

equivalently determined by the eigenvalues of the channel in the Bloch representation [5]. We denote

$$\lambda_X,\ \lambda_Y,\ \lambda_Z \in [-1,1]$$

the shrink factors of X, Y, Z under Λ. The physical error rate of Λ is $p := 1 - q_{\mathbb{I}}$.

Assumption 5.12 (Spatially homogeneous Pauli noise)**.** *We assume that, in the effective noise model of Corollary 4.20, all Twofold fibres in the patch $\mathcal{P}_L$ are subject to the* same *single-qubit Pauli channel Λ (the spatially homogeneous specialization of the local channels Λ_j), with shrink factors $(\lambda_X, \lambda_Y, \lambda_Z)$ and physical error rate $p = 1 - q_{\mathbb{I}}$. Thus the independent part of the global noise on the emergent register $\mathcal{H}_{\mathrm{reg}}^{(L)} \cong (\mathbb{C}^2)^{\otimes n_L}$ is*

$$\Lambda^{\otimes n_L} : \mathcal{B}(\mathcal{H}_{\mathrm{reg}}^{(L)}) \to \mathcal{B}(\mathcal{H}_{\mathrm{reg}}^{(L)}), \tag{188}$$

as in homogeneous local-noise threshold analyses for surface codes [1, 22, 4, 5]. Correlated and leakage contributions are accounted for separately via $\mathcal{N}_{\mathrm{corr}}^{(n_L)}$ and $\mathcal{N}_{\mathrm{leak}}^{(n_L)}$ in Corollary 4.20.

Stabilizer weights and robustness functional

Definition 5.13 (Pauli weight decomposition of stabilizers)**.** For each stabilizer $g \in S_L \subset \mathcal{PU}(\mathcal{H}_{\mathrm{reg}}^{(L)})$, write

$$g = \bigotimes_{j=1}^{n_L} P_j,$$

with each $P_j \in \{\mathbb{I}, X, Y, Z, -X, -Y, -Z\}$ acting on the Twofold fibre $\mathcal{H}_{(0,1),x_j}$. Ignoring global phases, we define:

$$w_X(g) := \#\{j : P_j \in \{X, -X\}\}, \tag{189}$$

$$w_Y(g) := \#\{j : P_j \in \{Y, -Y\}\}, \tag{190}$$

$$w_Z(g) := \#\{j : P_j \in \{Z, -Z\}\}, \tag{191}$$

and the total non-identity weight

$$w(g) := w_X(g) + w_Y(g) + w_Z(g). \tag{192}$$

This decomposes stabilizers according to their Pauli content, similarly to weight enumerators used in classical and quantum coding theory [19, 13, 5].

Definition 5.14 (Stabilizer robustness polynomial)**.** Let Λ be a single-qubit Pauli channel as in Definition 5.11, and let $\Lambda^{\otimes n_L}$ act on $\mathcal{H}_{\mathrm{reg}}^{(L)}$ as in Assumption 5.12. For any normalized code state $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$ (the surface-code subspace of Definition 5.3), define the *surface-code robustness functional*

$$F_{\mathrm{SC},L}(\Lambda) := \left\langle \psi_{\mathrm{SC},L} \middle| \Lambda^{\otimes n_L}(P_{\mathrm{SC},L}) \middle| \psi_{\mathrm{SC},L} \right\rangle. \tag{193}$$

Equivalently, using the stabilizer expansion of the projector, $P_{\mathrm{SC},L} = \frac{1}{|S_L|}\sum_{g\in S_L} g$, we introduce the *stabilizer robustness polynomial*

$$W_L(\lambda_X, \lambda_Y, \lambda_Z) := \frac{1}{|S_L|} \sum_{g \in S_L} \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)}. \tag{194}$$

This polynomial encodes, in a compact generating-function form, the distribution of Pauli weights across stabilizers and governs the response of the code subspace to homogeneous Pauli noise, in close analogy with weight-enumerator techniques in stabilizer-code analyses [19, 13, 5]. In particular, in the purely homogeneous Pauli-noise regime of Assumption 5.12 one has $F_{\mathrm{SC},L}(\Lambda) = W_L(\lambda_X, \lambda_Y, \lambda_Z)$, independent of the particular choice of code state $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$.

Lemma 5.15 (Evaluation of robustness functional via robustness polynomial)**.** *Let Λ be a single-qubit Pauli channel as in Definition 5.11, and let $\Lambda^{\otimes n_L}$ act homogeneously on the emergent Twofold register $\mathcal{H}_{\mathrm{reg}}^{(L)}$ as in Assumption 5.12. Then, for any normalized code state $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$, the robustness functional $F_{\mathrm{SC},L}(\Lambda)$ of (193) satisfies*

$$F_{\mathrm{SC},L}(\Lambda) = W_L(\lambda_X, \lambda_Y, \lambda_Z), \tag{195}$$

where W_L is the stabilizer robustness polynomial defined in Definition 5.14. In particular, $F_{\mathrm{SC},L}(\Lambda)$ is independent of the choice of $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$.

Proof. We proceed in three steps.

Step 1: Action of the Pauli channel on operators.

By Definition 5.11, the channel Λ is given by

$$\Lambda(\rho) = \sum_{P \in \{\mathbb{I}, X, Y, Z\}} q_P \, P \rho P, \qquad q_P \geq 0, \ \sum_P q_P = 1. \tag{196}$$

Although ρ is described as a state, the right-hand side defines a *linear* map on the full operator space $\mathcal{B}(\mathbb{C}^2)$: for any operator $O \in \mathcal{B}(\mathbb{C}^2)$ we can set

$$\Lambda(O) := \sum_P q_P \, POP. \tag{197}$$

Thus the same expression defines the Schrödinger action on states and the Heisenberg action on observables; in particular, Λ is self-adjoint with respect to the Hilbert–Schmidt inner product, as is standard for Pauli channels [13, 5].

By Definition 5.11, the Pauli operators $\{X, Y, Z\}$ are eigen-operators of Λ:

$$\Lambda(P) = \lambda_P P, \qquad P \in \{X, Y, Z\}, \tag{198}$$

with shrink factors $\lambda_P \in [-1, 1]$, and $\Lambda(\mathbb{I}) = \mathbb{I}$.

Now consider an n_L-qubit Pauli operator

$$g = \bigotimes_{j=1}^{n_L} P_j,$$

with each $P_j \in \{\mathbb{I}, X, Y, Z, -X, -Y, -Z\}$. Ignoring the overall phase ± 1, we can write $P_j = \pm \tilde{P}_j$ with $\tilde{P}_j \in \{\mathbb{I}, X, Y, Z\}$. The homogeneous channel $\Lambda^{\otimes n_L}$ acts as

$$\Lambda^{\otimes n_L}(g) = \bigotimes_{j=1}^{n_L} \Lambda(P_j) = \bigotimes_{j=1}^{n_L} \Lambda(\pm \tilde{P}_j) = \bigotimes_{j=1}^{n_L} \Big(\pm \Lambda(\tilde{P}_j)\Big) = \left(\prod_{j=1}^{n_L} \lambda_{\tilde{P}_j}\right) g, \tag{199}$$

where we define $\lambda_{\mathbb{I}} := 1$ for convenience.

Using the weight decomposition of Definition 5.13,

$$\begin{aligned} w_X(g) &:= \#\{j : P_j \in \{X, -X\}\}, \\ w_Y(g) &:= \#\{j : P_j \in \{Y, -Y\}\}, \\ w_Z(g) &:= \#\{j : P_j \in \{Z, -Z\}\}, \end{aligned}$$

we can rewrite the product of shrink factors as

$$\prod_{j=1}^{n_L} \lambda_{\tilde{P}_j} = \lambda_X^{w_X(g)}\, \lambda_Y^{w_Y(g)}\, \lambda_Z^{w_Z(g)}. \tag{200}$$

Hence, for every Pauli operator g (including all stabilizers),

$$\Lambda^{\otimes n_L}(g) = \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)}\, g. \tag{201}$$

Step 2: Applying $\Lambda^{\otimes n_L}$ to the stabilizer projector.

Recall that the surface-code projector at size L is

$$P_{\mathrm{SC},L} = \frac{1}{|S_L|} \sum_{g \in S_L} g, \tag{202}$$

where S_L is the stabilizer group of the code, as in Definition 5.3. By linearity of $\Lambda^{\otimes n_L}$ and using (201), we obtain

$$\begin{aligned} \Lambda^{\otimes n_L}(P_{\mathrm{SC},L}) &= \Lambda^{\otimes n_L}\left(\frac{1}{|S_L|} \sum_{g \in S_L} g\right) && (203) \\ &= \frac{1}{|S_L|} \sum_{g \in S_L} \Lambda^{\otimes n_L}(g) && (204) \\ &= \frac{1}{|S_L|} \sum_{g \in S_L} \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)}\, g. && (205) \end{aligned}$$

Step 3: Expectation in a code state and identification with W_L.

Let $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$ be any normalized code state. By definition of the code subspace as the joint $+1$ eigenspace of all stabilizers,

$$g\,|\psi_{\mathrm{SC},L}\rangle = |\psi_{\mathrm{SC},L}\rangle \qquad \text{for all } g \in S_L, \tag{206}$$

and thus

$$\left\langle \psi_{\mathrm{SC},L} \middle| g \middle| \psi_{\mathrm{SC},L} \right\rangle = 1 \qquad \text{for all } g \in S_L. \tag{207}$$

Using the definition (193) of the robustness functional and the expression above for

$\Lambda^{\otimes n_L}(P_{\mathrm{SC},L})$, we find

$$F_{\mathrm{SC},L}(\Lambda) = \left\langle \psi_{\mathrm{SC},L} \middle| \Lambda^{\otimes n_L}(P_{\mathrm{SC},L}) \middle| \psi_{\mathrm{SC},L} \right\rangle \tag{208}$$

$$= \left\langle \psi_{\mathrm{SC},L} \middle| \frac{1}{|S_L|} \sum_{g \in S_L} \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)} \, g \middle| \psi_{\mathrm{SC},L} \right\rangle \tag{209}$$

$$= \frac{1}{|S_L|} \sum_{g \in S_L} \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)} \left\langle \psi_{\mathrm{SC},L} \middle| g \middle| \psi_{\mathrm{SC},L} \right\rangle \tag{210}$$

$$= \frac{1}{|S_L|} \sum_{g \in S_L} \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)}. \tag{211}$$

But the last expression is precisely the stabilizer robustness polynomial $W_L(\lambda_X, \lambda_Y, \lambda_Z)$ defined in Definition 5.14. Therefore

$$F_{\mathrm{SC},L}(\Lambda) = W_L(\lambda_X, \lambda_Y, \lambda_Z). \tag{212}$$

In particular, the right-hand side no longer depends on $|\psi_{\mathrm{SC},L}\rangle$, so $F_{\mathrm{SC},L}(\Lambda)$ is independent of the particular code state chosen in $\mathcal{C}_L$.

This completes the proof. □

Corollary 5.16 (Robustness under homogeneous depolarizing noise)**.** *Assume the spatially homogeneous Pauli noise model of Assumption 5.12, and let the single-qubit noise channel $\Lambda_{\mathrm{dep},p}$ on each Twofold fibre be the standard depolarizing channel with error probability $p \in [0,1]$:*

$$\Lambda_{\mathrm{dep},p}(\rho) := (1-p)\,\rho + \frac{p}{3}\Big(X\rho X + Y\rho Y + Z\rho Z\Big), \tag{213}$$

as in canonical local-noise models for stabilizer and surface codes [13, 1, 5, 4]. Denote by

$$F_{\mathrm{SC},L}(p) := F_{\mathrm{SC},L}(\Lambda_{\mathrm{dep},p})$$

the robustness functional of Definition 5.14 for the surface code at size L.

Then:

(a) The depolarizing channel has identical shrink factors

$$\lambda_X(p) = \lambda_Y(p) = \lambda_Z(p) =: \lambda(p) = 1 - \frac{4}{3}p,$$

and the robustness functional is given exactly by

$$F_{\mathrm{SC},L}(p) = W_L\Big(\lambda(p), \lambda(p), \lambda(p)\Big) = \frac{1}{|S_L|} \sum_{g \in S_L} \lambda(p)^{w(g)}, \tag{214}$$

where $w(g) = w_X(g) + w_Y(g) + w_Z(g)$ is the non-identity weight of g.

(b) For small p one has the linear expansion

$$F_{\mathrm{SC},L}(p) = 1 - c_L\, p + O(p^2), \tag{215}$$

with

$$c_L = \frac{4}{3|S_L|} \sum_{g \in S_L} w(g), \tag{216}$$

where the coefficient c_L depends only on the stabilizer weight distribution of the code at size L (and not on the particular code state).

Proof. **(a) Shrink factors and exact expression.**

We first compute the action of the single-qubit depolarizing channel $\Lambda_{\mathrm{dep},p}$ on the Pauli operators. By definition,

$$\Lambda_{\mathrm{dep},p}(\rho) = (1-p)\,\rho + \frac{p}{3}\Big(X\rho X + Y\rho Y + Z\rho Z\Big). \tag{217}$$

Set $\rho = X$ and use the Pauli multiplication rules ($XX = \mathbb{I}$, $YX = iZ$, $ZX = -iY$, etc.) to compute:

$$\Lambda_{\mathrm{dep},p}(X) = (1-p)\,X + \frac{p}{3}\Big(XXX + YXY + ZXZ\Big). \tag{218}$$

Now,

$$XXX = X, \qquad YXY = -X, \qquad ZXZ = -X,$$

so

$$\begin{aligned}\Lambda_{\mathrm{dep},p}(X) &= (1-p)\,X + \frac{p}{3}\Big(X - X - X\Big) &(219)\\ &= (1-p)\,X - \frac{p}{3}X = \Big(1 - \frac{4}{3}p\Big)X. &(220)\end{aligned}$$

Thus the shrink factor for X is $\lambda_X(p) = 1 - \frac{4}{3}p$. By symmetry of the depolarizing channel under conjugation by Clifford operations (it is isotropic in the Bloch sphere), the same holds for Y and Z:

$$\lambda_Y(p) = \lambda_Z(p) = \lambda_X(p) =: \lambda(p) = 1 - \frac{4}{3}p.$$

Hence

$$\lambda_X(p) = \lambda_Y(p) = \lambda_Z(p) = \lambda(p).$$

Now apply Lemma 5.15, which states that for any Pauli-diagonal single-qubit channel Λ with shrink factors $(\lambda_X, \lambda_Y, \lambda_Z)$, one has

$$F_{\mathrm{SC},L}(\Lambda) = W_L(\lambda_X, \lambda_Y, \lambda_Z) = \frac{1}{|S_L|} \sum_{g \in S_L} \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)}.$$

Specializing this to the depolarizing case with $\lambda_X = \lambda_Y = \lambda_Z = \lambda(p)$ yields

$$\begin{aligned}F_{\mathrm{SC},L}(p) &= W_L\Big(\lambda(p), \lambda(p), \lambda(p)\Big) &(221)\\ &= \frac{1}{|S_L|} \sum_{g \in S_L} \lambda(p)^{w_X(g)+w_Y(g)+w_Z(g)} &(222)\\ &= \frac{1}{|S_L|} \sum_{g \in S_L} \lambda(p)^{w(g)}, &(223)\end{aligned}$$

establishing (214).

(b) Small-p expansion and coefficient c_L.

Define

$$\lambda(p) = 1 - \frac{4}{3}p,$$

and note that for each stabilizer $g \in S_L$, its contribution to $F_{\mathrm{SC},L}(p)$ is

$$\lambda(p)^{w(g)} = \left(1 - \frac{4}{3}p\right)^{w(g)}.$$

For small p, we expand this as a Taylor series around $p = 0$:

$$\lambda(p)^{w(g)} = \left(1 - \tfrac{4}{3}p\right)^{w(g)} \tag{224}$$

$$= 1 - \frac{4}{3}w(g)\,p + O(p^2), \tag{225}$$

where the $O(p^2)$ term is uniform in g for fixed L because $w(g) \le n_L$.

Substituting into (214) gives

$$F_{\mathrm{SC},L}(p) = \frac{1}{|S_L|}\sum_{g\in S_L}\left(1 - \frac{4}{3}w(g)p + O(p^2)\right) \tag{226}$$

$$= \frac{1}{|S_L|}\sum_{g\in S_L} 1 \;-\; \frac{4}{3}p\,\frac{1}{|S_L|}\sum_{g\in S_L} w(g) \;+\; \frac{1}{|S_L|}\sum_{g\in S_L} O(p^2). \tag{227}$$

Since there are $|S_L|$ terms in the first sum,

$$\frac{1}{|S_L|}\sum_{g\in S_L} 1 = 1.$$

The last term is still $O(p^2)$, because $|S_L|$ is finite for fixed L and we are averaging over g:

$$\frac{1}{|S_L|}\sum_{g\in S_L} O(p^2) = O(p^2).$$

Therefore

$$F_{\mathrm{SC},L}(p) = 1 - \frac{4}{3|S_L|}\left(\sum_{g\in S_L} w(g)\right)p + O(p^2). \tag{228}$$

Defining

$$c_L := \frac{4}{3|S_L|}\sum_{g\in S_L} w(g), \tag{229}$$

we recover the expansion (215):

$$F_{\mathrm{SC},L}(p) = 1 - c_L p + O(p^2).$$

By construction, c_L depends only on the stabilizer group S_L (through the weights $w(g)$) and not on any particular code state $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$, completing the proof. □

5.3.1 General stabilizer robustness and stabilizer witnesses

In this subsection we abstract the surface-code robustness functional introduced in Definition 5.14 to a general stabilizer setting, in the spirit of the stabilizer formalism for quantum error-correcting codes [19, 13, 5]. This will allow us to treat the surface-code family, GHZ and cluster ancillas, and other stabilizer resources within a single unified framework, including the cluster-state based constructions used in topological measurement-based schemes [11, 15, 5].

Definition 5.17 (General stabilizer robustness functional)**.** Let $n \in \mathbb{N}$ and let $S \subset \mathcal{P}_n$ be an abelian subgroup of the n-qubit Pauli group $\mathcal{P}_n$ with $-I \notin S$. Denote by

$$P_S \;:=\; \frac{1}{|S|} \sum_{g \in S} g \tag{230}$$

the projector onto the joint $+1$ eigenspace of S, and let $\mathcal{C}_S := \mathrm{im}(P_S) \subset (\mathbb{C}^2)^{\otimes n}$ be the corresponding stabilizer code space [13, 5]. Let $\Lambda : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2)$ be a single-qubit CPTP channel and consider its n-fold tensor product $\Lambda^{\otimes n} : \mathcal{B}\Big((\mathbb{C}^2)^{\otimes n}\Big) \to \mathcal{B}\Big((\mathbb{C}^2)^{\otimes n}\Big)$.

The *general stabilizer robustness functional* associated with S and Λ is defined by

$$F_S(\Lambda) \;:=\; \mathrm{Tr}\Big[P_S\, \Lambda^{\otimes n}(P_S)\Big]. \tag{231}$$

Equivalently, if $|\Psi_S\rangle \in \mathcal{C}_S$ is any (normalized) stabilizer state with $g\,|\Psi_S\rangle = |\Psi_S\rangle$ for all $g \in S$, then

$$F_S(\Lambda) \;=\; \langle \Psi_S|\; \Lambda^{\otimes n}\Big(|\Psi_S\rangle\langle\Psi_S|\Big)\; |\Psi_S\rangle\,, \tag{232}$$

so that $F_S(\Lambda)$ can be interpreted as a stability / fidelity functional for the stabilizer resource under local noise [13, 5].

Proposition 5.18 (Polynomial structure of $F_S(\Lambda)$)**.** *Suppose that Λ is a single-qubit Pauli-covariant channel as in Definition 5.11, i.e.*

$$\Lambda(P) \;=\; \lambda_P\, P \qquad \text{for } P \in \{\mathbb{I}, X, Y, Z\}, \tag{233}$$

with real "shrink factors" $\lambda_X, \lambda_Y, \lambda_Z \in [-1, 1]$ and $\Lambda(\mathbb{I}) = \mathbb{I}$, as in standard Pauli-noise models for fault-tolerant QEC [1, 5]. For each $g \in S$ write g in tensor-product Pauli form and denote by $w_P(g)$ the number of tensor factors equal to $P \in \{X, Y, Z\}$.

Assume that S is a full-rank stabilizer group on n qubits (so that S defines a unique stabilizer state, i.e. an $[[n, 0]]$ code), as in the stabilizer-resource constructions used for GHZ, cluster, and surface-code resource states in the book [13, 5]. Then

$$F_S(\Lambda) \;=\; \frac{1}{|S|} \sum_{g \in S} \lambda_X^{w_X(g)}\, \lambda_Y^{w_Y(g)}\, \lambda_Z^{w_Z(g)}. \tag{234}$$

In particular, $F_S(\Lambda)$ is a real polynomial in $(\lambda_X, \lambda_Y, \lambda_Z)$ whose coefficients are determined entirely by the stabilizer weight distribution of S, in analogy with weight-enumerator techniques for stabilizer codes [19, 13, 5].

Proof. Because S is a full-rank stabilizer group on n qubits (an $[[n,0]]$ stabilizer code), its joint $+1$ eigenspace $\mathcal{C}_S$ is one-dimensional. Let $|\Psi_S\rangle$ be the unique (normalized) stabilizer state with $g\,|\Psi_S\rangle = |\Psi_S\rangle$ for all $g \in S$. In this regime the projector onto $\mathcal{C}_S$ is simply

$$P_S \;=\; |\Psi_S\rangle\langle\Psi_S|\,. \tag{235}$$

On the other hand, by the standard stabilizer-state identity one also has

$$P_S \;=\; \frac{1}{|S|}\sum_{g\in S} g, \tag{236}$$

with $|S| = 2^n$ in the full-rank ($[[n,0]]$) case [13, 5].

By definition (cf. Definition 5.17), the stabilizer robustness functional may be written as

$$F_S(\Lambda) \;=\; \langle\Psi_S|\;\Lambda^{\otimes n}\Big(|\Psi_S\rangle\langle\Psi_S|\Big)\;|\Psi_S\rangle\,. \tag{237}$$

(Equivalently, $F_S(\Lambda) = \mathrm{Tr}[P_S\,\Lambda^{\otimes n}(P_S)]$, but in the $[[n,0]]$ case these coincide because P_S has rank 1.)

Using the stabilizer expansion of P_S we rewrite

$$\begin{aligned} F_S(\Lambda) &= \langle\Psi_S|\;\Lambda^{\otimes n}\Big(\frac{1}{|S|}\sum_{g\in S} g\Big)\;|\Psi_S\rangle \\ &= \frac{1}{|S|}\sum_{g\in S}\langle\Psi_S|\;\Lambda^{\otimes n}(g)\;|\Psi_S\rangle\,. \end{aligned} \tag{238}$$

Now use Pauli covariance of Λ. Any $g \in S$ can be written as

$$g = \bigotimes_{j=1}^{n} P_j, \qquad P_j \in \{\pm\mathbb{I}, \pm X, \pm Y, \pm Z\}.$$

Because Λ is Pauli-diagonal with shrink factors $(\lambda_X, \lambda_Y, \lambda_Z)$ and $\Lambda(\mathbb{I}) = \mathbb{I}$, we have for each single-qubit Pauli

$$\Lambda(P_j) = \lambda_{P_j}\,P_j,$$

where by convention $\lambda_{\pm P} = \lambda_P$ and $\lambda_{\mathbb{I}} = 1$. Therefore

$$\Lambda^{\otimes n}(g) = \bigotimes_{j=1}^{n}\Lambda(P_j) = \Big(\prod_{j=1}^{n}\lambda_{P_j}\Big)\,g = \lambda_X^{w_X(g)}\lambda_Y^{w_Y(g)}\lambda_Z^{w_Z(g)}\,g. \tag{239}$$

Substituting (239) into (238) gives

$$F_S(\Lambda) = \frac{1}{|S|}\sum_{g\in S}\lambda_X^{w_X(g)}\lambda_Y^{w_Y(g)}\lambda_Z^{w_Z(g)}\,\langle\Psi_S|\,g\,|\Psi_S\rangle\,. \tag{240}$$

But $|\Psi_S\rangle$ is by construction a $+1$ eigenstate of every $g \in S$, so $g\,|\Psi_S\rangle = |\Psi_S\rangle$ and hence $\langle\Psi_S|\,g\,|\Psi_S\rangle = 1$ for all $g \in S$. Thus

$$F_S(\Lambda) = \frac{1}{|S|}\sum_{g\in S}\lambda_X^{w_X(g)}\lambda_Y^{w_Y(g)}\lambda_Z^{w_Z(g)}, \tag{241}$$

which is exactly eq. (234). This exhibits $F_S(\Lambda)$ as a real polynomial in $(\lambda_X, \lambda_Y, \lambda_Z)$ whose coefficients are given by the stabilizer-weight distribution $\{w_X(g), w_Y(g), w_Z(g)\}_{g\in S}$, as claimed. $\square$

Definition 5.19 (Stabilizer witness family)**.** Let $S \subset \mathcal{P}_n$ and P_S be as in Definition 5.17. For any real parameter $\alpha \in \mathbb{R}$ we define the associated *stabilizer witness*

$$W_\alpha \;:=\; \alpha\,\mathbb{I} - P_S. \tag{242}$$

Given a noisy output state

$$\rho_\Lambda \;:=\; \Lambda^{\otimes n}\Big(|\Psi_S\rangle\langle\Psi_S|\Big), \tag{243}$$

where $|\Psi_S\rangle$ is any stabilizer state in $\mathcal{C}_S$, we write

$$\Big\langle W_\alpha \Big\rangle_\Lambda \;:=\; \mathrm{Tr}\Big[W_\alpha\,\rho_\Lambda\Big] \tag{244}$$

for the expectation value of W_α on ρ_Λ, in analogy with entanglement / resource witnesses in quantum information theory [5].

Lemma 5.20. *With notation as in Definitions 5.17 and 5.19, assume that S is a full-rank stabilizer group on n qubits, so that the code space $\mathcal{C}_S$ is one-dimensional and*

$$P_S \;=\; |\Psi_S\rangle\langle\Psi_S| \tag{245}$$

for a unique (normalized) stabilizer state $|\Psi_S\rangle$. Let

$$\rho_\Lambda \;:=\; \Lambda^{\otimes n}\Big(|\Psi_S\rangle\langle\Psi_S|\Big),$$

and let $W_\alpha := \alpha\,\mathbb{I} - P_S$ be the associated stabilizer witness. Then

$$\Big\langle W_\alpha \Big\rangle_\Lambda \;=\; \alpha - F_S(\Lambda), \tag{246}$$

where $F_S(\Lambda)$ is the stabilizer robustness functional of Definition 5.17.

Proof. By definition of the witness W_α and the noisy state ρ_Λ,

$$\begin{aligned}\Big\langle W_\alpha \Big\rangle_\Lambda &:= \mathrm{Tr}\Big[W_\alpha\,\rho_\Lambda\Big] = \mathrm{Tr}\Big[(\alpha\,\mathbb{I} - P_S)\,\rho_\Lambda\Big] \\ &= \alpha\;\mathrm{Tr}(\rho_\Lambda) \;-\; \mathrm{Tr}\Big[P_S\,\rho_\Lambda\Big].\end{aligned} \tag{247}$$

Since Λ is CPTP, ρ_Λ is a normalized state and $\mathrm{Tr}(\rho_\Lambda) = 1$, so the first term in (247) simplifies to α:

$$\alpha\;\mathrm{Tr}(\rho_\Lambda) \;=\; \alpha. \tag{248}$$

Thus

$$\Big\langle W_\alpha \Big\rangle_\Lambda \;=\; \alpha - \mathrm{Tr}\Big[P_S\,\rho_\Lambda\Big] \;=\; \alpha - \mathrm{Tr}\Big[P_S\,\Lambda^{\otimes n}\Big(|\Psi_S\rangle\langle\Psi_S|\Big)\Big]. \tag{249}$$

Now we use the fact that, in the full-rank stabilizer-state setting considered here, the projector P_S is rank one:

$$P_S \;=\; |\Psi_S\rangle\langle\Psi_S|\,. \tag{250}$$

By Definition 5.17, the stabilizer robustness functional is

$$F_S(\Lambda) \;:=\; \mathrm{Tr}\Big[P_S\,\Lambda^{\otimes n}(P_S)\Big]. \tag{251}$$

Substituting $P_S = |\Psi_S\rangle\langle\Psi_S|$ into this definition, we obtain

$$\begin{aligned} F_S(\Lambda) &= \mathrm{Tr}\Big[|\Psi_S\rangle\langle\Psi_S|\ \Lambda^{\otimes n}\Big(|\Psi_S\rangle\langle\Psi_S|\Big)\Big] \\ &= \langle\Psi_S|\ \Lambda^{\otimes n}\Big(|\Psi_S\rangle\langle\Psi_S|\Big)\ |\Psi_S\rangle\,. \end{aligned} \tag{252}$$

But the right-hand side of (252) is exactly the second term in (249):

$$\mathrm{Tr}\Big[P_S\,\Lambda^{\otimes n}\Big(|\Psi_S\rangle\langle\Psi_S|\Big)\Big] = F_S(\Lambda). \tag{253}$$

Inserting this into (249) yields

$$\Big\langle W_\alpha \Big\rangle_\Lambda \;=\; \alpha - F_S(\Lambda), \tag{254}$$

which is the claimed identity. □

Corollary 5.21 (Polynomial witness thresholds for stabilizer resources)**.** *Assume $S \subset \mathcal{P}_n$ is a full-rank stabilizer group on n qubits, so that the corresponding code space $\mathcal{C}_S$ is one-dimensional and $P_S = |\Psi_S\rangle\langle\Psi_S|$ for a unique (normalized) stabilizer state $|\Psi_S\rangle$, as in the standard stabilizer-state setting [13, 5]. Let $\mathcal{F} \subset \mathcal{S}\Big((\mathbb{C}^2)^{\otimes n}\Big)$ be a convex set of "free" states (e.g. separable or biseparable states) and define*

$$\alpha_{\max} \;:=\; \sup_{\sigma\in\mathcal{F}} \langle\Psi_S|\,\sigma\,|\Psi_S\rangle\,. \tag{255}$$

Set $W_{\alpha_{\max}} := \alpha_{\max}\mathbb{I} - P_S$. Then, for any Pauli-covariant channel Λ as in Definition 5.11, the following hold:

(i) For all $\sigma \in \mathcal{F}$ one has

$$\mathrm{Tr}\Big[W_{\alpha_{\max}}\sigma\Big] \;\geq\; 0. \tag{256}$$

(ii) If the robustness functional satisfies

$$F_S(\Lambda) \;>\; \alpha_{\max}, \tag{257}$$

then the noisy stabilizer state $\rho_\Lambda = \Lambda^{\otimes n}(|\Psi_S\rangle\langle\Psi_S|)$ is not contained in $\mathcal{F}$ and

$$\mathrm{Tr}\Big[W_{\alpha_{\max}}\rho_\Lambda\Big] \;<\; 0. \tag{258}$$

In particular, $W_{\alpha_{\max}}$ is an entanglement/resource witness for the stabilizer resource defined by S, and the witness-detection region is the polynomial domain

$$\Big\{(\lambda_X,\lambda_Y,\lambda_Z) : F_S(\Lambda(\lambda_X,\lambda_Y,\lambda_Z)) > \alpha_{\max}\Big\} \subset [-1,1]^3, \tag{259}$$

with F_S as in eq. (234).

Proof. (1) For any $\sigma \in \mathcal{F}$ we compute

$$\mathrm{Tr}\Big[W_{\alpha_{\max}}\sigma\Big] \;=\; \mathrm{Tr}\Big[(\alpha_{\max}\mathbb{I} - P_S)\sigma\Big] \;=\; \alpha_{\max}\,\mathrm{Tr}(\sigma) - \mathrm{Tr}(P_S\sigma). \tag{260}$$

Since σ is a state, $\mathrm{Tr}(\sigma) = 1$, and because $P_S = |\Psi_S\rangle\langle\Psi_S|$ in the full-rank stabilizer-state setting,

$$\mathrm{Tr}(P_S\sigma) = \langle\Psi_S|\,\sigma\,|\Psi_S\rangle\,. \tag{261}$$

Hence

$$\mathrm{Tr}\Big[W_{\alpha_{\max}}\sigma\Big] = \alpha_{\max} - \langle\Psi_S|\,\sigma\,|\Psi_S\rangle \;\geq\; 0, \tag{262}$$

by the definition of $\alpha_{\max}$ as the supremum of $\langle\Psi_S|\,\sigma\,|\Psi_S\rangle$ over $\sigma \in \mathcal{F}$.

(2) For the noisy stabilizer state $\rho_\Lambda := \Lambda^{\otimes n}(|\Psi_S\rangle\langle\Psi_S|)$, we use Lemma 5.20, which applies in the rank-one (stabilizer-state) case and gives, for any real α,

$$\Big\langle W_\alpha \Big\rangle_\Lambda := \mathrm{Tr}\Big[W_\alpha\rho_\Lambda\Big] = \alpha - F_S(\Lambda), \tag{263}$$

with $F_S(\Lambda)$ the stabilizer robustness functional of Definition 5.17. Setting $\alpha = \alpha_{\max}$ we obtain

$$\mathrm{Tr}\Big[W_{\alpha_{\max}}\rho_\Lambda\Big] = \alpha_{\max} - F_S(\Lambda). \tag{264}$$

If $F_S(\Lambda) > \alpha_{\max}$ as in eq. (257), then

$$\mathrm{Tr}\Big[W_{\alpha_{\max}}\rho_\Lambda\Big] \;<\; 0. \tag{265}$$

By part (1), every $\sigma \in \mathcal{F}$ yields a non-negative expectation value for $W_{\alpha_{\max}}$, so a state with negative expectation value cannot lie in $\mathcal{F}$. Thus $\rho_\Lambda \notin \mathcal{F}$, and $W_{\alpha_{\max}}$ acts as an entanglement/resource witness for the stabilizer resource $|\Psi_S\rangle$ in the usual sense of witness theory in quantum information [5]. The description of the witness-detection region in terms of the polynomial $F_S(\Lambda)$ then follows directly from the polynomial expression eq. (234). ☐

Remark 5.22 (Special cases)**.** For later reference we note that:

- Choosing $S = S_L$ to be the surface-code stabilizer group on a patch $\mathcal{P}_L$ recovers the surface-code robustness functional $F_S(\Lambda) = F_{\mathrm{SC},L}(\Lambda)$ of Definition 5.14, with S_L as in the standard planar/rotated surface codes [16, 21, 1, 4, 15, 5].
- Choosing $S = S_{\mathrm{GHZ}}$ to be the GHZ stabilizer group yields $F_S(\Lambda) = F_{\mathrm{GHZ}}(\Lambda)$, and similarly for small cluster-state stabilizers (e.g. S_{C_4} for a four-qubit cluster state) we recover the polynomials $F_{C_4}(\Lambda)$ used in the ancilla analysis of Section 8.3, in direct analogy with cluster-state based topological and measurement-based schemes [11, 12, 15, 5].

Thus all stabilizer-based robustness polynomials appearing in this article arise as instances of the general construction in Definition 5.17 and Proposition 5.18. In particular, for the GHZ and small cluster-state stabilizers we are in the rank-one (stabilizer-state) case $P_S = |\Psi_S\rangle\langle\Psi_S|$ used in Corollary 5.21, whereas for the surface-code family S_L one has a higher-dimensional code space but the polynomial expression for $F_S(\Lambda)$ in Proposition 5.18 still applies verbatim.

Proposition 5.23 (Expression for $F_{\mathrm{SC},L}(\Lambda)$)**.** *Under Assumption 5.12, the functional* $F_{\mathrm{SC},L}(\Lambda)$ *is independent of the particular encoded code state* $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$*, and*

$$F_{\mathrm{SC},L}(\Lambda) = W_L(\lambda_X, \lambda_Y, \lambda_Z) = \frac{1}{|S_L|} \sum_{g \in S_L} \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)}. \tag{266}$$

Proof. By definition of the surface-code projector

$$P_{\mathrm{SC},L} = \frac{1}{|S_L|} \sum_{g \in S_L} g,$$

the code space $\mathcal{C}_L$ is the joint $+1$ eigenspace of all stabilizers $g \in S_L$ [16, 21, 1, 4, 15, 5]. Hence for any normalized code state $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$,

$$g\,|\psi_{\mathrm{SC},L}\rangle = |\psi_{\mathrm{SC},L}\rangle \qquad \text{for all } g \in S_L, \tag{267}$$

and therefore

$$P_{\mathrm{SC},L}|\psi_{\mathrm{SC},L}\rangle = \left(\frac{1}{|S_L|} \sum_{g \in S_L} g\right)|\psi_{\mathrm{SC},L}\rangle = |\psi_{\mathrm{SC},L}\rangle. \tag{268}$$

By Definition 5.14, the robustness functional is

$$F_{\mathrm{SC},L}(\Lambda) = \left\langle \psi_{\mathrm{SC},L} \middle| \Lambda^{\otimes n_L}(P_{\mathrm{SC},L}) \middle| \psi_{\mathrm{SC},L} \right\rangle, \tag{269}$$

with $\Lambda^{\otimes n_L}$ the homogeneous single-qubit Pauli channel on $\mathcal{H}_{\mathrm{reg}}^{(L)}$ from Assumption 5.12. Expanding $P_{\mathrm{SC},L}$ gives

$$\begin{aligned} F_{\mathrm{SC},L}(\Lambda) &= \left\langle \psi_{\mathrm{SC},L} \middle| \Lambda^{\otimes n_L}(P_{\mathrm{SC},L}) \middle| \psi_{\mathrm{SC},L} \right\rangle \\ &= \frac{1}{|S_L|} \sum_{g \in S_L} \left\langle \psi_{\mathrm{SC},L} \middle| \Lambda^{\otimes n_L}(g) \middle| \psi_{\mathrm{SC},L} \right\rangle. \end{aligned} \tag{270}$$

Write each stabilizer $g \in S_L$ in tensor-product Pauli form

$$g = \bigotimes_{j=1}^{n_L} P_j,$$

with $P_j \in \{\pm\mathbb{I}, \pm X, \pm Y, \pm Z\}$, and let $w_X(g), w_Y(g), w_Z(g)$ be the Pauli weights as in Definition 5.13. Since Λ is Pauli-diagonal (Pauli-covariant) as in Definition 5.11, we have

$$\Lambda(P_j) = \lambda_{P_j} P_j,$$

where $\lambda_{\pm P} = \lambda_P$ and $\lambda_{\mathbb{I}} = 1$. Hence

$$\Lambda^{\otimes n_L}(g) = \bigotimes_{j=1}^{n_L} \Lambda(P_j) = \bigotimes_{j=1}^{n_L} \lambda_{P_j} P_j = \left(\prod_{j=1}^{n_L} \lambda_{P_j}\right) g = \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)}\, g. \tag{271}$$

Therefore,

$$\begin{aligned} \left\langle \psi_{\mathrm{SC},L} \middle| \Lambda^{\otimes n_L}(g) \middle| \psi_{\mathrm{SC},L} \right\rangle &= \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)} \left\langle \psi_{\mathrm{SC},L} \middle| g \middle| \psi_{\mathrm{SC},L} \right\rangle \\ &= \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)}, \end{aligned} \tag{272}$$

since $g|\psi_{\mathrm{SC},L}\rangle = |\psi_{\mathrm{SC},L}\rangle$ for all $g \in S_L$.

Substituting back into the expression for $F_{\mathrm{SC},L}(\Lambda)$, we obtain

$$F_{\mathrm{SC},L}(\Lambda) = \frac{1}{|S_L|} \sum_{g \in S_L} \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)} = W_L(\lambda_X, \lambda_Y, \lambda_Z), \tag{273}$$

which is manifestly independent of the particular encoded state $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$. This is exactly the claimed polynomial expression (266), in agreement with the stabilizer-code picture of surface codes under homogeneous Pauli noise [1, 4, 5]. □

Weight moments and decay of the robustness functional

Definition 5.24 (Weight distribution and moments)**.** For each L, let G_L be a random stabilizer drawn uniformly from S_L:

$$\mathbb{P}(G_L = g) = \frac{1}{|S_L|}, \quad g \in S_L,$$

in line with the standard stabilizer-code picture of treating S_L as a finite abelian Pauli group [19, 13, 5]. Define the random variables

$$W_X := w_X(G_L), \quad W_Y := w_Y(G_L), \quad W_Z := w_Z(G_L), \quad W := W_X + W_Y + W_Z.$$

We denote the mean weight

$$\bar{w}_L := \mathbb{E}[W] = \frac{1}{|S_L|} \sum_{g \in S_L} w(g),$$

and, more generally, for $k \geq 1$,

$$\mathbb{E}[W^k] = \frac{1}{|S_L|} \sum_{g \in S_L} w(g)^k.$$

These moments describe the stabilizer weight distribution of the code at size L, analogous to classical and quantum weight-enumerator data for stabilizer codes [19, 13, 5].

Lemma 5.25 (Robustness functional as a generating function)**.** *In the isotropic case $\lambda_X = \lambda_Y = \lambda_Z = \lambda \in [0,1]$ (e.g. an effective depolarizing-type Pauli channel on each physical qubit, as commonly assumed in surface-code threshold analyses [1, 22, 4, 5]), the robustness functional can be written as*

$$F_{\mathrm{SC},L}(\lambda) := F_{\mathrm{SC},L}(\Lambda_\lambda) = \frac{1}{|S_L|} \sum_{g \in S_L} \lambda^{w(g)} = \mathbb{E}\Big[\lambda^W\Big]. \tag{274}$$

Moreover, for all $\lambda \in [0,1]$,

$$\lambda^{\bar{w}_L} \ \leq\ F_{\mathrm{SC},L}(\lambda) \ \leq\ 1, \tag{275}$$

and the derivatives of $F_{\mathrm{SC},L}(\lambda)$ at $\lambda = 1$ encode the moments of W:

$$F_{\mathrm{SC},L}(1) = 1, \tag{276}$$

$$F'_{\mathrm{SC},L}(1) = \mathbb{E}[W] = \bar{w}_L, \tag{277}$$

$$F''_{\mathrm{SC},L}(1) = \mathbb{E}[W(W-1)]. \tag{278}$$

Proof. Under Assumption 5.12 and in the isotropic case $\lambda_X = \lambda_Y = \lambda_Z = \lambda$, Proposition 5.23 gives

$$F_{\mathrm{SC},L}(\Lambda_\lambda) = \frac{1}{|S_L|} \sum_{g \in S_L} \lambda_X^{w_X(g)} \lambda_Y^{w_Y(g)} \lambda_Z^{w_Z(g)} = \frac{1}{|S_L|} \sum_{g \in S_L} \lambda^{w_X(g)+w_Y(g)+w_Z(g)} = \frac{1}{|S_L|} \sum_{g \in S_L} \lambda^{w(g)}.$$

By Definition 5.24, this is exactly

$$F_{\mathrm{SC},L}(\lambda) = \mathbb{E}[\lambda^W],$$

with $W = w(G_L)$ for a uniformly random $G_L \in S_L$. This proves the generating function identity.

For $\lambda \in [0,1]$ and $w \geq 0$ we have $0 \leq \lambda^w \leq 1$, so

$$0 \leq F_{\mathrm{SC},L}(\lambda) = \mathbb{E}[\lambda^W] \leq 1,$$

giving the upper bound. (The lower bound 0 is trivial; the sharper bound $\lambda^{\bar{w}_L}$ will follow from convexity.)

For $0 < \lambda < 1$, consider $f(w) := \lambda^w$ for $w \geq 0$. Writing $\lambda = e^{\ln \lambda}$ with $\ln \lambda < 0$, we have

$$f''(w) = (\ln \lambda)^2 \, \lambda^w > 0,$$

so f is convex. By Jensen's inequality,

$$\mathbb{E}[\lambda^W] = \mathbb{E}[f(W)] \geq f(\mathbb{E}[W]) = \lambda^{\bar{w}_L},$$

which yields the desired lower bound for $0 < \lambda < 1$. The case $\lambda = 1$ is trivially $F_{\mathrm{SC},L}(1) = 1 = \lambda^{\bar{w}_L}$, and the case $\lambda = 0$ follows by continuity of the polynomial $F_{\mathrm{SC},L}(\lambda)$ in $\lambda \in [0,1]$.

For the derivatives, note that $F_{\mathrm{SC},L}(\lambda)$ is a finite polynomial in λ:

$$F_{\mathrm{SC},L}(\lambda) = \sum_{w \geq 0} p_w \lambda^w,$$

where $p_w := \mathbb{P}(W = w)$ and only finitely many p_w are nonzero (since S_L is finite). Thus we may differentiate termwise:

$$F'_{\mathrm{SC},L}(\lambda) = \sum_{w \geq 0} p_w \, w \, \lambda^{w-1}, \tag{279}$$

$$F''_{\mathrm{SC},L}(\lambda) = \sum_{w \geq 0} p_w \, w(w-1) \, \lambda^{w-2}. \tag{280}$$

Evaluating at $\lambda = 1$ gives

$$F_{\mathrm{SC},L}(1) = \sum_{w \geq 0} p_w = 1, \tag{281}$$

$$F'_{\mathrm{SC},L}(1) = \sum_{w \geq 0} p_w \, w = \mathbb{E}[W] = \bar{w}_L, \tag{282}$$

$$F''_{\mathrm{SC},L}(1) = \sum_{w \geq 0} p_w \, w(w-1) = \mathbb{E}[W(W-1)]. \tag{283}$$

This identifies the first and second derivatives at $\lambda = 1$ with the first factorial moment and the second factorial moment of the stabilizer weight distribution, exactly as in weight-enumerator generating-function methods for stabilizer codes [19, 13, 5]. □

Remark 5.26. Lemma 5.25 shows that $F_{\mathrm{SC},L}$ acts as the (factorial) moment-generating function of the stabilizer weight distribution, evaluated at $\log \lambda$:

$$F_{\mathrm{SC},L}(\lambda) = \mathbb{E}[\lambda^W] = \mathbb{E}\Big[e^{(\log \lambda)\, W}\Big]$$

for $\lambda > 0$. The derivatives at $\lambda = 1$ thus encode the moments $\mathbb{E}[W^k]$ (via the usual relations between factorial and ordinary moments). For $\lambda < 1$, the lower bound $F_{\mathrm{SC},L}(\lambda) \geq \lambda^{\bar{w}_L}$ expresses that the robustness decays at least as fast as $\lambda^{\bar{w}_L}$ with decreasing λ. In practice, the presence of low-weight stabilizers limits how sharply $F_{\mathrm{SC},L}$ can decay, but for large codes the bulk of S_L consists of high-weight elements, so $\bar{w}_L$ typically grows with n_L, consistent with the growing stabilizer weight scales in surface-code families used for threshold analyses [1, 4, 5].

5.4 Logical Error Scaling and Threshold Behaviour

We now connect the effective local Pauli noise model of Corollary 4.20 to standard surface-code threshold results for local noise [1, 22, 4, 15, 5], and translate these into the IQR valuation language developed in the Holo-State framework [8].

Physical noise model on the emergent register

From Corollary 4.20, the emergent register $\mathcal{H}_{\mathrm{reg}}^{(L)}$ at size L is subject, per QEC cycle, to an effective noise channel of the form

$$\mathcal{N}_{\mathrm{eff}}^{(n_L)} = \left(\bigotimes_{x_j \in \mathcal{P}_L} \Lambda_j \right) + \mathcal{N}_{\mathrm{corr}}^{(n_L)} + \mathcal{N}_{\mathrm{leak}}^{(n_L)}, \tag{284}$$

where:

- each Λ_j is a single-qubit Pauli channel on $\mathcal{H}_{(0,1),x_j}$ with local error rate $p^{(j)} := 1 - q_{\mathbb{I}}^{(j)} \leq p$ for some uniform $p > 0$, as in the local Pauli-noise regime for surface-code thresholds [1, 22, 4, 5];
- $\|\mathcal{N}_{\mathrm{corr}}^{(n_L)}\|_\diamond \leq \epsilon_{\mathrm{corr}}(n_L)$, with $\epsilon_{\mathrm{corr}}(n_L) = O(n_L e^{-r/\xi})$ controlled by the correlation length ξ, modelling weakly correlated noise in the spirit of more general threshold results [2, 3, 5];
- $\|\mathcal{N}_{\mathrm{leak}}^{(n_L)}\|_\diamond = O(\varepsilon_{\mathrm{ps}})$, where $\varepsilon_{\mathrm{ps}}$ is the leakage parameter of Definition 3.15, capturing excursions outside the computational subspace [5].

Assumption 5.27 (Threshold-compatible noise regime)**.** *We assume that the parameters p, $\epsilon_{\mathrm{corr}}(n_L)$, and $\varepsilon_{\mathrm{ps}}$ satisfy*

$$p < p_*, \qquad \epsilon_{\mathrm{corr}}(n_L) \ll 1, \qquad \varepsilon_{\mathrm{ps}} \ll 1, \tag{285}$$

for some constant $p_* > 0$ independent of L, and that $\epsilon_{\mathrm{corr}}(n_L)$ remains bounded (or decays) as L grows (e.g. for fixed correlation length ξ and increasing patch size), in line with standard threshold-compatible local-noise assumptions for topological codes [1, 22, 4, 15, 5].

Surface-code threshold under local Pauli noise

Theorem 5.28 (Surface-code logical error scaling under local noise)**.** *Let $\{\mathcal{C}_L\}$ be the family of surface codes defined in Definition 5.3, with code distances $d_L \sim L$, and let the effective physical noise per QEC cycle on $\mathcal{H}^{(L)}_{\mathrm{reg}}$ be given by $\mathcal{N}^{(n_L)}_{\mathrm{eff}}$ as above, satisfying Assumption 5.27.*

Then there exist constants $p_* > 0$, $c > 0$, and $C, C_{\mathrm{corr}}, C_{\mathrm{leak}} > 0$, independent of L, such that for all $p < p_*$ the worst-case logical error probability per cycle (on the encoded qubit in $\mathcal{C}_L$) obeys

$$p_L^{(L)} \;\le\; C\left(\frac{p}{p_*}\right)^{c\,d_L} + C_{\mathrm{corr}}\,\epsilon_{\mathrm{corr}}(n_L) + C_{\mathrm{leak}}\,\varepsilon_{\mathrm{ps}}, \tag{286}$$

where $p_L^{(L)}$ is the worst-case logical error probability defined in Definition 2.7.

Sketch of proof. In the idealized case of strictly independent local Pauli noise (no correlated tail and no leakage), Theorem 5.28 reduces to the known surface-code threshold theorem: below some critical physical error rate p_*, the logical error probability for an $L \times L$ surface code decays exponentially in $d_L \sim L$; see, e.g., the original and subsequent analyses of toric/surface codes and their thresholds [16, 1, 22, 4, 15, 5]. In that setting one has a bound of the form

$$p_L^{(L)} \;\le\; C\left(\frac{p}{p_*}\right)^{cd_L}$$

for suitable constants $c, C > 0$ independent of L.

We now treat $\mathcal{N}^{(n_L)}_{\mathrm{corr}}$ and $\mathcal{N}^{(n_L)}_{\mathrm{leak}}$ as perturbations of the independent Pauli part. The correlated part satisfies $\|\mathcal{N}^{(n_L)}_{\mathrm{corr}}\|_\diamond \le \epsilon_{\mathrm{corr}}(n_L)$, so its contribution to the worst-case logical failure probability per cycle is bounded by a term of order $\epsilon_{\mathrm{corr}}(n_L)$, with a constant factor absorbed into C_{corr}, as in perturbative and cluster-expansion treatments of weakly correlated noise [2, 3, 5].

Similarly, the leakage term $\mathcal{N}^{(n_L)}_{\mathrm{leak}}$ has diamond norm $O(\varepsilon_{\mathrm{ps}})$ by the Phase 1 analysis of the leakage parameter (Definition 3.15 and Lemma 3.19), and thus induces at most an additive logical error contribution of size $C_{\mathrm{leak}}\varepsilon_{\mathrm{ps}}$, as in standard leakage-noise models for QEC [5] (events in which the state exits the code subspace or re-enters in an uncontrolled way).

Combining the exponential suppression from the independent local-Pauli part with these two perturbative contributions yields the bound (286). More rigorous treatments of thresholds under weakly correlated noise and leakage follow similar lines, using cluster expansions and quantitative bounds on the influence of non-local and leakage errors in the general fault-tolerance frameworks [2, 17, 3, 5]. ☐

Translation to IQR Thesis valuation

Recall from Definition 2.9 that the logical Thesis valuation is

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) := 1 - p_L^{(L)}.$$

Corollary 5.29 (Logical Thesis valuation and distance)**.** *Under the hypotheses of Theorem 5.28, there exist constants $p_* > 0$, $c > 0$, and $C, C_{\mathrm{corr}}, C_{\mathrm{leak}} > 0$ independent of L such that, for all*

$p < p_*$, the Thesis valuation of the logical correctness proposition $\varphi_L^{(L)}$ satisfies

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) \;=\; 1 - p_L^{(L)} \;\geq\; 1 - C\Big(\frac{p}{p_*}\Big)^{c\,d_L} - C_{\mathrm{corr}}\,\epsilon_{\mathrm{corr}}(n_L) - C_{\mathrm{leak}}\,\varepsilon_{\mathrm{ps}}. \tag{287}$$

In particular, for fixed noise parameters $p < p_*$, $\epsilon_{\mathrm{corr}}(n_L) \ll 1$ and $\varepsilon_{\mathrm{ps}} \ll 1$, one can make $v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)})$ arbitrarily close to 1 by choosing L (and hence d_L) sufficiently large, reproducing the asymptotic reliability of topological stabilizer codes in the IQR valuation language [16, 1, 4, 5, 8].

Proof. By definition (Definition 2.9),

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) = 1 - p_L^{(L)}, \tag{288}$$

where $p_L^{(L)}$ is the worst-case logical error probability per QEC cycle on the encoded qubit in $\mathcal{C}_L$.

Theorem 5.28 states that, under the threshold-compatible noise assumptions of Assumption 5.27, there exist constants $p_* > 0$, $c > 0$, and $C, C_{\mathrm{corr}}, C_{\mathrm{leak}} > 0$ independent of L such that, for all $p < p_*$,

$$p_L^{(L)} \;\leq\; C\left(\frac{p}{p_*}\right)^{c\,d_L} + C_{\mathrm{corr}}\,\epsilon_{\mathrm{corr}}(n_L) + C_{\mathrm{leak}}\,\varepsilon_{\mathrm{ps}}. \tag{289}$$

Substituting (289) into $v_{\mathrm{Th}} = 1 - p_L^{(L)}$ yields the lower bound

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) = 1 - p_L^{(L)} \;\geq\; 1 - C\Big(\frac{p}{p_*}\Big)^{c\,d_L} - C_{\mathrm{corr}}\,\epsilon_{\mathrm{corr}}(n_L) - C_{\mathrm{leak}}\,\varepsilon_{\mathrm{ps}}, \tag{290}$$

which is exactly (287).

For the asymptotic statement, fix noise parameters with $p < p_*$ and choose $\epsilon_{\mathrm{corr}}(n_L)$ and $\varepsilon_{\mathrm{ps}}$ sufficiently small (as required by Assumption 5.27; in particular, $\epsilon_{\mathrm{corr}}(n_L)$ remains bounded or decays with increasing L for fixed correlation length ξ [1, 4, 5]). For any target accuracy $\delta > 0$, choose L large enough that

$$C\Big(\frac{p}{p_*}\Big)^{cd_L} \;<\; \frac{\delta}{2}, \tag{291}$$

which is possible since $(p/p_*)^{cd_L}$ decays exponentially in $d_L \sim L$ for fixed $p < p_*$ [16, 1, 4, 5]. Then choose the noise parameters so that

$$C_{\mathrm{corr}}\,\epsilon_{\mathrm{corr}}(n_L) + C_{\mathrm{leak}}\,\varepsilon_{\mathrm{ps}} \;<\; \frac{\delta}{2}. \tag{292}$$

Combining these estimates with (287) gives

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) \;\geq\; 1 - \delta. \tag{293}$$

Since $\delta > 0$ was arbitrary, this shows that $v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) \to 1$ as $d_L \to \infty$ for fixed subthreshold noise parameters, reproducing the standard asymptotic reliability of topological stabilizer codes in the IQR valuation language [16, 1, 4, 5, 8]. □

Remark 5.30. Corollary 5.29 realizes the core logical part of the IQR threshold programme: under suitable noise assumptions derived from the Fourfold correlator (Phase 2) and controlled leakage (Phase 1), the Thesis valuation of the logical correctness proposition approaches 1 as the code distance grows, i.e. the emergent Twofold register supports asymptotically reliable quantum information processing [1, 4, 5, 8]. In later phases, this logical robustness will be combined with holosymmetric process-equivalence valuations to obtain a full IQR fault-tolerance threshold theorem, in analogy with the way standard threshold theorems combine logical accuracy with fault-tolerant circuit constructions [2, 17, 3, 5].

6 Phase 4: Process Layer – Holosymmetry for Syndrome Cycles

In this section we refine the process-level part of the IQR threshold construction. For each code size L, we consider two different full QEC schedules ("gadgets") implementing the same logical operation and use the holosymmetric dual-history interferometer to compare them. Under effective local Pauli noise (Phase 2), we obtain explicit expressions for the holosymmetric process overlap and derive a process-level threshold condition on a differential noise parameter δ, in direct analogy with fault-tolerant gadget and schedule comparisons in standard QEC and topological-code frameworks [13, 1, 2, 3, 4, 5, 8].

Throughout we assume the ambient IQR setting and surface-code family described in Phases 0–3 [1, 4, 5, 8].

6.1 Dual-History Interferometer for a Full QEC Cycle

We first specialize the dual-history interferometer of Section 2 to the case of a full QEC cycle for the surface-code family at size L, in the spirit of coherent fault-tolerant gadget comparisons in the stabilizer formalism [13, 2, 3, 5].

Definition 6.1 (QEC schedules for a full cycle at size L)**.** Fix a code size L and corresponding surface-code patch $\mathcal{C}_L \subset \mathcal{H}_{\mathrm{reg}}^{(L)}$ with ideal logical channel $\mathcal{N}_{\mathrm{ideal}}^{(L)}$ (Definition 2.4).

A *QEC schedule for one full cycle at size* L is a concrete implementation of:

- full stabilizer measurement on $\mathcal{C}_L$ (syndrome extraction);
- classical decoding of the syndrome;
- application of the corresponding correction operations on the physical qubits.

Such schedules encompass the usual syndrome-measurement, decoding, and correction routines in stabilizer and surface-code QEC [13, 1, 4, 5].

We model a given schedule A as a CPTP map

$$\mathcal{N}_A^{(L)} : \mathcal{S}(\mathcal{H}_{\mathrm{em}}^{(L)}) \to \mathcal{S}(\mathcal{H}_{\mathrm{em}}^{(L)}), \tag{294}$$

arising from a sequence of unitaries and measurements in the emergent GNS Hilbert space $\mathcal{H}_{\mathrm{em}}^{(L)}$ (including ancillas and environment), followed by tracing out non-logical degrees of freedom, in line with the IQR/Holo-State realization of physical processes [8] and the general operator-sum description of QEC gadgets [13, 5]. Similarly, a second schedule B is modelled by a CPTP map

$$\mathcal{N}_B^{(L)} : \mathcal{S}(\mathcal{H}_{\rm em}^{(L)}) \to \mathcal{S}(\mathcal{H}_{\rm em}^{(L)}). \tag{295}$$

We assume that, in the ideal (noise-free) limit, both schedules implement the same logical channel $\mathcal{N}_{\rm ideal}^{(L)}$ on $\mathcal{C}_L$, as in standard notions of logically equivalent fault-tolerant gadgets [2, 3, 5].

Remark 6.2. Concretely, one may think of $\mathcal{N}_A^{(L)}$ and $\mathcal{N}_B^{(L)}$ as generated by ideal coherent QEC unitaries $\hat{\mathbf{U}}_A^{(L)}$ and $\hat{\mathbf{U}}_B^{(L)}$ in the GNS arena, followed by effective noise maps (e.g. Pauli channels on the emergent register) arising from couplings encoded in the Fourfold correlator $\hat{G}_\Lambda$ [8], in direct analogy with noise-dressed fault-tolerant gadgets in the circuit model [2, 3, 5].

Definition 6.3 (Dual-history interferometer for a full cycle)**.** Let $\mathcal{N}_A^{(L)}$ and $\mathcal{N}_B^{(L)}$ be two QEC schedules for size L as in Definition 6.1. The *dual-history holosymmetric interferometer* for a full cycle at size L is defined as follows:

(a) Introduce a reference qubit R and prepare it in the state

$$|+\rangle_R := \frac{1}{\sqrt{2}}(|0\rangle_R + |1\rangle_R),$$

as in standard interferometric comparisons of alternative quantum processes [5].

(b) Prepare an encoded logical state $\rho_{\log} \in \mathcal{S}(\mathcal{C}_L)$ and embed it into $\mathcal{H}_{\rm em}^{(L)}$ via the isometry $V_L : \mathcal{C}_L \hookrightarrow \mathcal{H}_{\rm em}^{(L)}$, giving the initial joint state

$$\rho_{\rm in}^{(L)} := |+\rangle\langle+|_R \otimes V_L \rho_{\log} V_L^\dagger. \tag{296}$$

(c) Choose Kraus representations $\mathcal{N}_A^{(L)}(\sigma) = \sum_i K_{A,i}\sigma K_{A,i}^\dagger$ and $\mathcal{N}_B^{(L)}(\sigma) = \sum_j K_{B,j}\sigma K_{B,j}^\dagger$. Define the controlled schedule

$$\mathcal{N}_{\rm ctrl}^{(L)}(\cdot) := \sum_i L_i(\cdot)L_i^\dagger + \sum_j M_j(\cdot)M_j^\dagger, \tag{297}$$

with Kraus operators

$$L_i := |0\rangle\langle 0|_R \otimes K_{A,i}, \qquad M_j := |1\rangle\langle 1|_R \otimes K_{B,j}. \tag{298}$$

This defines a CPTP map on $\mathcal{S}(\mathcal{H}_R \otimes \mathcal{H}_{\rm em}^{(L)})$ whose marginal action on the system conditioned on $R = 0$ (resp. $R = 1$) is $\mathcal{N}_A^{(L)}$ (resp. $\mathcal{N}_B^{(L)}$), and is independent of the particular Kraus decompositions chosen for $\mathcal{N}_A^{(L)}$ and $\mathcal{N}_B^{(L)}$. Apply this channel to obtain

$$\tilde{\rho}^{(L)} := \mathcal{N}_{\rm ctrl}^{(L)}\left(\rho_{\rm in}^{(L)}\right). \tag{299}$$

(d) Apply a Hadamard gate H_R on the reference qubit:

$$\rho_{\rm out}^{(L)} := (H_R \otimes \mathbb{I}_{\rm em})\,\tilde{\rho}^{(L)}\,(H_R \otimes \mathbb{I}_{\rm em})^\dagger. \tag{300}$$

(e) Measure the reference in the computational basis and define the *holosymmetric effect*

$$E_{\mathrm{Holo}} := |0\rangle\langle 0|_R \otimes \mathbb{I}_{\mathrm{em}}, \tag{301}$$

so that the interference pattern observed on R directly probes the coherent comparison between the two QEC histories in the IQR setting [8].

Definition 6.4 (Holosymmetric process valuation for one cycle)**.** The *holosymmetric process valuation* (Thesis valuation in the process corner) for the comparison of schedules A and B at size L is

$$v_{\mathrm{Holo}}^{(L)} := \mathrm{Tr}\Big(E_{\mathrm{Holo}}\, \rho_{\mathrm{out}}^{(L)}\Big), \tag{302}$$

interpreted as the IQR holosymmetric analogue of an interferometric overlap / visibility between two fault-tolerant QEC gadgets [5, 8]. Equivalently, we define the *holosymmetric process overlap*

$$C^{(L)} := 2\, v_{\mathrm{Holo}}^{(L)} - 1, \tag{303}$$

so that

$$v_{\mathrm{Holo}}^{(L)} = \frac{1}{2}\Big(1 + C^{(L)}\Big). \tag{304}$$

Lemma 6.5 (Holosymmetric overlap vs logical process fidelity)**.** *Let $\mathcal{N}_A^{(L)}$ and $\mathcal{N}_B^{(L)}$ be two QEC schedules at size L as in Definition 6.1, and let $\mathcal{N}_{A,\mathrm{log}}^{(L)}$ and $\mathcal{N}_{B,\mathrm{log}}^{(L)}$ denote their restrictions to the logical code space $\mathcal{C}_L$ (obtained by compressing to $\mathcal{C}_L$ with $P_{\mathrm{SC},L}$ and tracing out non-logical degrees of freedom).*

Consider the dual-history interferometer of Definition 6.3 with logical input taken to be a maximally entangled logical state

$$|\Phi_{\mathrm{log}}^{+}\rangle := \frac{1}{\sqrt{2}}\Big(|0_L\rangle \otimes |0_L\rangle + |1_L\rangle \otimes |1_L\rangle\Big) \in \mathcal{C}_L \otimes \mathcal{C}_L,$$

embedded into the emergent space by $V_L \otimes \mathbb{I}$. Then the holosymmetric process overlap $C^{(L)}$ of Definition 6.4 can be written as

$$C^{(L)} \;=\; 2v_{\mathrm{Holo}}^{(L)} - 1 \;=\; \mathrm{Re}\, F_{\mathrm{proc}}\Big(\mathcal{N}_{A,\mathrm{log}}^{(L)}, \mathcal{N}_{B,\mathrm{log}}^{(L)}\Big), \tag{305}$$

where

$$F_{\mathrm{proc}}\Big(\mathcal{N}_{A,\mathrm{log}}^{(L)}, \mathcal{N}_{B,\mathrm{log}}^{(L)}\Big) := \mathrm{Tr}\Big[J\Big(\mathcal{N}_{A,\mathrm{log}}^{(L)}\Big)^{\dagger} J\Big(\mathcal{N}_{B,\mathrm{log}}^{(L)}\Big)\Big] \Big/ d_{\mathrm{log}}^{2} \tag{306}$$

is the (normalized) Hilbert–Schmidt overlap of the Choi matrices of the two logical channels, with $d_{\mathrm{log}} = 2$ for a single encoded logical qubit [5]. Here

$$J\Big(\mathcal{N}_{X,\mathrm{log}}^{(L)}\Big) := \Big(\mathrm{id} \otimes \mathcal{N}_{X,\mathrm{log}}^{(L)}\Big)\Big(|\Phi_{\mathrm{log}}^{+}\rangle\langle\Phi_{\mathrm{log}}^{+}|\Big), \qquad X \in \{A, B\}, \tag{307}$$

is the (unnormalized) Choi matrix of the restricted logical channel, as in the standard Choi–Jamiołkowski representation for quantum channels [5].

Proof. We first recast the schedules in Stinespring form, in line with the IQR/Holo-State description of physical processes [8]. For $X \in \{A, B\}$, fix an environment Hilbert space $\mathcal{H}_E^{(L)}$ and a pure reference environment state $|0_E\rangle$, and choose Stinespring isometries

$$U_X^{(L)} : \mathcal{H}_{\mathrm{em}}^{(L)} \otimes \mathcal{H}_E^{(L)} \;\to\; \mathcal{H}_{\mathrm{em}}^{(L)} \otimes \mathcal{H}_E^{(L)} \tag{308}$$

such that

$$\mathcal{N}_X^{(L)}(\sigma) = \mathrm{Tr}_E\Big[U_X^{(L)}\Big(\sigma \otimes |0_E\rangle\langle 0_E|\Big)U_X^{(L)\dagger}\Big]. \tag{309}$$

Compressing to the logical code space with $P_{\mathrm{SC},L}$ and tracing out non-logical degrees of freedom yields the restricted logical channels $\mathcal{N}_{X,\log}^{(L)}$ acting on $\mathcal{S}(\mathcal{C}_L)$.

Next, we embed the maximally entangled logical input $|\Phi_{\log}^+\rangle$ into the emergent setting via $V_L \otimes \mathbb{I}$ and include the environment. The total initial state is

$$\rho_{\mathrm{in}}^{(L)} := |+\rangle\langle +|_R \otimes |\Phi_{\log}^+\rangle\langle \Phi_{\log}^+|_{LL'} \otimes |0_E\rangle\langle 0_E|\,, \tag{310}$$

where L denotes the code block subjected to the schedule and L' is a logical reference system.

Define the controlled Stinespring unitary

$$U_{\mathrm{ctrl}}^{(L)} := |0\rangle\langle 0|_R \otimes U_A^{(L)} + |1\rangle\langle 1|_R \otimes U_B^{(L)}. \tag{311}$$

This implements the two QEC schedules coherently in the two "history" branches controlled by R, in accordance with the dual-history picture of the IQR framework [8]. The post-controlled state is

$$\tilde{\rho}^{(L)} := U_{\mathrm{ctrl}}^{(L)}\,\rho_{\mathrm{in}}^{(L)}\,U_{\mathrm{ctrl}}^{(L)\dagger}. \tag{312}$$

After applying a Hadamard H_R to R we obtain

$$\rho_{\mathrm{out}}^{(L)} := (H_R \otimes \mathbb{I})\,\tilde{\rho}^{(L)}\,(H_R \otimes \mathbb{I})^\dagger, \tag{313}$$

and the holosymmetric valuation is

$$v_{\mathrm{Holo}}^{(L)} := \mathrm{Tr}\Big[\big(\,|0\rangle\langle 0|_R \otimes \mathbb{I}_{LL'E}\,\big)\rho_{\mathrm{out}}^{(L)}\Big] \tag{314}$$

as in Definition 6.4.

A standard interferometric calculation (see, e.g., the process-overlap discussion in [5]) shows that, for any pair of Stinespring isometries $U_A^{(L)}$, $U_B^{(L)}$ and pure input $|\Psi_{\mathrm{in}}\rangle \in \mathcal{H}_{\mathrm{em}}^{(L)} \otimes \mathcal{H}_E^{(L)}$, the above construction yields

$$v_{\mathrm{Holo}}^{(L)} = \frac{1}{2}\Big(1 + \mathrm{Re}\,\langle \Psi_{\mathrm{in}}|U_B^{(L)\dagger}U_A^{(L)}|\Psi_{\mathrm{in}}\rangle\Big). \tag{315}$$

In our case $|\Psi_{\mathrm{in}}\rangle = |\Phi_{\log}^+\rangle_{LL'} \otimes |0_E\rangle$, so the holosymmetric overlap $C^{(L)} := 2v_{\mathrm{Holo}}^{(L)} - 1$ is

$$C^{(L)} = \mathrm{Re}\,\langle \Phi_{\log}^+, 0_E|U_B^{(L)\dagger}U_A^{(L)}|\Phi_{\log}^+, 0_E\rangle\,. \tag{316}$$

On the other hand, the Choi matrix of the restricted logical channel $\mathcal{N}_{X,\log}^{(L)}$ is

$$J\Big(\mathcal{N}_{X,\log}^{(L)}\Big) = \Big(\mathrm{id} \otimes \mathcal{N}_{X,\log}^{(L)}\Big)\Big(|\Phi_{\log}^+\rangle\langle \Phi_{\log}^+|\Big), \tag{317}$$

which, in terms of the Stinespring isometry, can be written as

$$J\Big(\mathcal{N}^{(L)}_{X,\log}\Big) = \mathrm{Tr}_E\Big[(\mathbb{I}_{L'} \otimes U_X^{(L)})\Big(|\Phi^+_{\log}\rangle\langle\Phi^+_{\log}|_{LL'} \otimes |0_E\rangle\langle 0_E|\Big)(\mathbb{I}_{L'} \otimes U_X^{(L)\dagger})\Big]. \tag{318}$$

A straightforward Hilbert–Schmidt inner-product calculation then yields

$$\mathrm{Tr}\Big[J\Big(\mathcal{N}^{(L)}_{A,\log}\Big)^\dagger J\Big(\mathcal{N}^{(L)}_{B,\log}\Big)\Big] = d^2_{\log}\ \langle\Phi^+_{\log}, 0_E|U_B^{(L)\dagger}U_A^{(L)}|\Phi^+_{\log}, 0_E\rangle\,, \tag{319}$$

with $d_{\log} = \dim\mathcal{C}_L = 2$ for the single-encoded-qubit surface-code family [1, 4, 5]. Combining this with (316) and taking the real part gives

$$C^{(L)} = \mathrm{Re}\,\frac{\mathrm{Tr}\Big[J\Big(\mathcal{N}^{(L)}_{A,\log}\Big)^\dagger J\Big(\mathcal{N}^{(L)}_{B,\log}\Big)\Big]}{d^2_{\log}} = \mathrm{Re}\,F_{\mathrm{proc}}\Big(\mathcal{N}^{(L)}_{A,\log}, \mathcal{N}^{(L)}_{B,\log}\Big), \tag{320}$$

and hence $v^{(L)}_{\mathrm{Holo}} = \frac{1}{2}(1 + C^{(L)})$ as in Definition 6.4. This establishes the claimed relation between the holosymmetric process valuation and the logical process fidelity, in direct analogy with standard channel-overlap interferometry in quantum information theory [5]. □

We now relate the dual-history interferometer of Definition 6.3 to the induced logical channels on the surface code and introduce a quantitative parameter for the *process difference* between two QEC schedules at size L.

Logical channels induced by schedules

Definition 6.6 (Logical channels for schedules at size L)**.** Let $\mathcal{C}_L \subset \mathcal{H}^{(L)}_{\mathrm{reg}}$ be the surface-code subspace at size L with projector $P_{\mathrm{SC},L}$ (Definition 5.3), and let $\mathcal{N}^{(L)}_A$ and $\mathcal{N}^{(L)}_B$ be two QEC schedules for one full cycle at size L as in Definition 6.1.

We define the corresponding *logical channels* on $\mathcal{C}_L$ by

$$\mathcal{N}^{(L)}_{A,\log} : \mathcal{S}(\mathcal{C}_L) \to \mathcal{S}(\mathcal{C}_L), \qquad \mathcal{N}^{(L)}_{A,\log}(\rho_{\log}) := \mathrm{Tr}_{\mathrm{anc}}\Big[P_{\mathrm{SC},L}\,\mathcal{N}^{(L)}_A\Big(V_L\rho_{\log}V_L^\dagger\Big)\,P_{\mathrm{SC},L}\Big], \tag{321}$$

and similarly

$$\mathcal{N}^{(L)}_{B,\log} : \mathcal{S}(\mathcal{C}_L) \to \mathcal{S}(\mathcal{C}_L), \qquad \mathcal{N}^{(L)}_{B,\log}(\rho_{\log}) := \mathrm{Tr}_{\mathrm{anc}}\Big[P_{\mathrm{SC},L}\,\mathcal{N}^{(L)}_B\Big(V_L\rho_{\log}V_L^\dagger\Big)\,P_{\mathrm{SC},L}\Big], \tag{322}$$

where $V_L : \mathcal{C}_L \hookrightarrow \mathcal{H}_{\mathrm{em}}$ is the embedding isometry and $\mathrm{Tr}_{\mathrm{anc}}$ denotes the partial trace over all non-logical degrees of freedom (ancillas, environment, etc.) in the IQR/Holo-State setting [13, 5, 8].

By assumption, in the noise-free limit one has

$$\mathcal{N}^{(L)}_{A,\log} = \mathcal{N}^{(L)}_{B,\log} = \mathcal{N}^{(L)}_{\mathrm{ideal}}$$

(Definition 2.4), in accordance with the standard notion of logically equivalent fault-tolerant gadgets [2, 3, 5].

Definition 6.7 (Differential logical noise parameter)**.** For a given size L and two schedules A, B as above, we define the *differential logical noise parameter*

$$\delta_{\text{log}}^{(L)} \;:=\; \frac{1}{2}\left\|\mathcal{N}_{A,\text{log}}^{(L)} - \mathcal{N}_{B,\text{log}}^{(L)}\right\|_{\diamond}, \tag{323}$$

where $\|\cdot\|_{\diamond}$ denotes the diamond norm on channels acting on the logical code space [5]. Thus $\delta_{\text{log}}^{(L)} \in [0,1]$ quantifies the operational distinguishability between the two logical processes per cycle at size L.

Holosymmetric valuation vs logical channel distance

We now relate the holosymmetric process valuation $v_{\text{Holo}}^{(L)}$ of Definition 6.4 to the distance between the induced logical channels.

Lemma 6.8 (Holosymmetric valuation and diamond distance)**.** *Let $\mathcal{N}_A^{(L)}$ and $\mathcal{N}_B^{(L)}$ be two QEC schedules for one cycle at size L as in Definition 6.1, and let $v_{\text{Holo}}^{(L)}$ be the corresponding holosymmetric process valuation (Definition 6.4). Then there exists a constant $C_{\text{Holo}} > 0$, depending only on the dimension of the logical code space (here $\dim \mathcal{C}_L = 2$), such that*

$$\left|v_{\text{Holo}}^{(L)} - v_{\text{Holo,ideal}}^{(L)}\right| \;\leq\; C_{\text{Holo}}\, \delta_{\text{log}}^{(L)}, \tag{324}$$

where $v_{\text{Holo,ideal}}^{(L)}$ is the holosymmetric valuation obtained when both branches implement the same ideal logical channel $\mathcal{N}_{\text{ideal}}^{(L)}$.

Proof. The dual-history interferometer of Definition 6.3 is constructed from standard quantum operations on the reference qubit and the embedded logical system, followed by the measurement of the holosymmetric effect E_{Holo} on the output state $\rho_{\text{out}}^{(L)}$. In particular, fixing the encoded input state ρ_{log} and the embedding V_L, the map

$$\left(\mathcal{N}_{A,\text{log}}^{(L)}, \mathcal{N}_{B,\text{log}}^{(L)}\right) \longmapsto v_{\text{Holo}}^{(L)}$$

is implemented by a fixed sequence of completely positive trace-preserving maps (dilations and partial traces) and a two-outcome POVM $\{E_{\text{Holo}}, \mathbb{I} - E_{\text{Holo}}\}$ on the final joint system (cf. the general IQR/Holo-State process-valuation construction [8]).

Let us regard the pair of channels $(\mathcal{N}_{A,\text{log}}^{(L)}, \mathcal{N}_{B,\text{log}}^{(L)})$ as a single CPTP map $\mathcal{T}^{(L)}$ acting on the logical input and the reference qubit (plus any ancillas), obtained by composing:

(i) preparation of $|+\rangle_R\langle+| \otimes \rho_{\text{log}}$;

(ii) conditional application of $\mathcal{N}_{A,\text{log}}^{(L)}$ or $\mathcal{N}_{B,\text{log}}^{(L)}$ under coherent control of R in the IQR/Holo-State sense [8];

(iii) a final Hadamard on R and the holosymmetric measurement.

This construction is linear in each argument and completely positive overall.

If we denote by $\mathcal{T}^{(L)}_{\mathrm{ideal}}$ the map obtained by replacing both logical channels with the ideal channel $\mathcal{N}^{(L)}_{\mathrm{ideal}}$, then the difference of the corresponding output statistics for the measurement $\{E_{\mathrm{Holo}}, \mathbb{I} - E_{\mathrm{Holo}}\}$ satisfies the standard channel-continuity bound

$$\left| v^{(L)}_{\mathrm{Holo}} - v^{(L)}_{\mathrm{Holo,ideal}} \right| \;\le\; \frac{1}{2} \left\| \mathcal{T}^{(L)} - \mathcal{T}^{(L)}_{\mathrm{ideal}} \right\|_{\diamond}, \tag{325}$$

since the left-hand side is the total variation distance between two binary outcome distributions obtained from channels $\mathcal{T}^{(L)}$ and $\mathcal{T}^{(L)}_{\mathrm{ideal}}$ applied to a fixed input state and a fixed POVM [5].

By construction, $\mathcal{T}^{(L)}$ is obtained from $\mathcal{N}^{(L)}_{A,\log}$ and $\mathcal{N}^{(L)}_{B,\log}$ by a constant-depth circuit of Stinespring dilations, tensoring with ancillas, and partial traces. Such operations are diamond-norm contractive (or at worst Lipschitz with a constant depending only on the finite ancilla dimension), so there exists $C_{\mathrm{Holo}} > 0$, depending only on the logical system dimension, such that

$$\left\| \mathcal{T}^{(L)} - \mathcal{T}^{(L)}_{\mathrm{ideal}} \right\|_{\diamond} \;\le\; C_{\mathrm{Holo}} \left\| \mathcal{N}^{(L)}_{A,\log} - \mathcal{N}^{(L)}_{B,\log} \right\|_{\diamond} = 2\, C_{\mathrm{Holo}}\, \delta^{(L)}_{\log}. \tag{326}$$

Combining the two inequalities yields

$$\left| v^{(L)}_{\mathrm{Holo}} - v^{(L)}_{\mathrm{Holo,ideal}} \right| \;\le\; C_{\mathrm{Holo}}\, \delta^{(L)}_{\log},$$

as claimed. □

Remark 6.9. In the special case where the holosymmetric interferometer is implemented using a maximally entangled logical input on $\mathcal{C}_L \otimes \mathcal{C}_L$ and the standard Choi–Jamiołkowski representation of channels, the overlap $C^{(L)} = 2v^{(L)}_{\mathrm{Holo}} - 1$ can be identified (up to dimension-dependent constants) with a normalized Hilbert–Schmidt inner product of the logical Choi matrices of $\mathcal{N}^{(L)}_{A,\log}$ and $\mathcal{N}^{(L)}_{B,\log}$, and Lemma 6.8 reduces to the usual inequalities relating diamond distance, Choi trace distance, and process fidelities [5]. The IQR holosymmetric construction [8] generalizes this picture to the dual-history Holo-State setting but obeys the same continuity bounds.

In later subsections we will combine:

- the logical error scaling from Theorem 5.28 (Phase 3);
- bounds of the form

$$\left\| \mathcal{N}^{(L)}_{S,\log} - \mathcal{N}^{(L)}_{\mathrm{ideal}} \right\|_{\diamond} = O\!\left(\left(\frac{p}{p_*}\right)^{c d_L} + \epsilon_{\mathrm{corr}}(n_L) + \varepsilon_{\mathrm{ps}} \right), \qquad S \in \{A, B\},$$

- the triangle inequality together with Lemma 6.8,

to obtain a *process-level* threshold corollary of the form

$$v^{(L)}_{\mathrm{Holo}} \;\ge\; 1 - C'_{\mathrm{Holo}} \left[\left(\frac{p}{p_*}\right)^{c d_L} + \epsilon_{\mathrm{corr}}(n_L) + \varepsilon_{\mathrm{ps}} + \delta_{\mathrm{phys}} \right], \tag{327}$$

where δ_{phys} encodes any additional schedule-dependent differential noise (e.g. mismatch in physical noise parameters for A and B), and C'_{Holo} is a constant independent of L.

6.2 Pauli-Channel Holosymmetry and Multi-Cycle Behaviour

We now specialize to the effective local Pauli noise regime of Phase 2 and derive closed-form expressions for $C^{(L)}$ and $v_{\text{Holo}}^{(L)}$ in terms of single-site error distributions, in direct analogy with Pauli-channel models used in threshold and surface-code analyses [18, 20, 1, 22, 4, 5, 8].

Pauli error distributions and single-site overlap

Assumption 6.10 (Independent Pauli noise per PQS for each schedule)**.** *For a single QEC cycle at size L, suppose that:*

- *Schedule A is implemented as an ideal coherent process $\hat{\mathbf{U}}_A^{(L)}$ on the emergent register and ancillas, followed by a layer of independent single-qubit Pauli noise on each PQS touched by the gadget.*
- *Schedule B is implemented as an ideal coherent process $\hat{\mathbf{U}}_B^{(L)}$ (with the same logical action) followed by an independent layer of (possibly different) Pauli noise on each touched PQS.*

Such independent-per-qubit Pauli noise models are standard in threshold and fault-tolerance analyses for stabilizer and surface codes [13, 1, 2, 3, 4, 5]. Let Q_L denote the set of PQS loci touched by either schedule within the cycle, and let $m_L := |Q_L|$. For each $x \in Q_L$, the noise in schedule A is a single-qubit Pauli channel $\Lambda_x^{(A)}$ with error probabilities

$$\mathbf{q}_x^{(A)} := \left\{ q_x^{(A)}(P) \right\}_{P \in \{\mathbb{I}, X, Y, Z\}}, \quad q_x^{(A)}(P) \geq 0, \ \sum_P q_x^{(A)}(P) = 1,$$

and similarly for schedule B with probabilities $\mathbf{q}_x^{(B)}$.

Definition 6.11 (Single-site Pauli overlap)**.** For each $x \in Q_L$, define the *single-site Pauli overlap*

$$k_x := \sum_{P \in \{\mathbb{I}, X, Y, Z\}} \sqrt{q_x^{(A)}(P)\, q_x^{(B)}(P)}. \tag{328}$$

If the noise is homogeneous across Q_L, we write $k_x \equiv k$ for all $x \in Q_L$.

Pauli-channel holosymmetry theorem for multi-qubit gadgets

Theorem 6.12 (Pauli-channel holosymmetry for a multi-qubit QEC gadget)**.** *Let Assumption 6.10 hold for schedules A and B at size L, and consider the dual-history interferometer of Definition 6.3. Then the holosymmetric process overlap $C^{(L)}$ factors as*

$$C^{(L)} = \prod_{x \in Q_L} k_x. \tag{329}$$

In particular, in the spatially homogeneous case $k_x \equiv k$ one has

$$C^{(L)} = k^{m_L}, \tag{330}$$

and therefore

$$v_{\text{Holo}}^{(L)} = \frac{1}{2}\Big(1 + k^{m_L}\Big). \tag{331}$$

Proof. We work in the interaction picture where the ideal coherent parts $\hat{\mathbf{U}}_A^{(L)}$ and $\hat{\mathbf{U}}_B^{(L)}$ have been stripped off and absorbed into a common ideal evolution; equivalently, we consider $C^{(L)}$ normalized by the ideal (noise-free) holosymmetric overlap, so that only the effect of the Pauli noise layers remains. This is standard in fault-tolerant gadget comparisons, where one studies the noise dressing of logically equivalent circuits [2, 3, 5].

For each $x \in Q_L$, choose a Kraus representation of $\Lambda_x^{(A)}$ and $\Lambda_x^{(B)}$ based on classical error flags. Concretely, introduce an orthonormal basis $\{|P\rangle_x : P \in \{\mathbb{I}, X, Y, Z\}\}$ for a local environment system E_x, and define

$$K_{x,P}^{(A)} := \sqrt{q_x^{(A)}(P)}\, P \otimes |P\rangle_x, \tag{332}$$

$$K_{x,P}^{(B)} := \sqrt{q_x^{(B)}(P)}\, P \otimes |P\rangle_x. \tag{333}$$

Then

$$\Lambda_x^{(S)}(\rho) = \sum_P K_{x,P}^{(S)}\, \rho\, K_{x,P}^{(S)\,\dagger}, \quad S \in \{A, B\},$$

realizes each single-qubit Pauli channel as a unitary coupling to an environment prepared in a fixed reference state $|e_0\rangle_x := \sum_P |P\rangle_x$ (up to normalization).

For the full gadget, the total noise channel in branch S is $\bigotimes_{x\in Q_L} \Lambda_x^{(S)}$, with Kraus operators labelled by error patterns $\mathbf{P} = (P_x)_{x\in Q_L}$:

$$K_{\mathbf{P}}^{(S)} := \bigotimes_{x\in Q_L} K_{x,P_x}^{(S)} = \Big(\prod_{x\in Q_L} \sqrt{q_x^{(S)}(P_x)}\Big)\Big(\bigotimes_{x\in Q_L} P_x\Big) \otimes \Big(\bigotimes_{x\in Q_L} |P_x\rangle_x\Big). \tag{334}$$

In the dual-history interferometer, the reference qubit R coherently selects branch A or B. After the controlled application of the noisy gadgets (in the interaction picture), and before the final Hadamard on R, the joint state of R and the rest of the system has the block form (suppressing normalization and the logical input for clarity)

$$\rho_{\text{mid}}^{(L)} = \frac{1}{2}\sum_{\mathbf{P},\mathbf{P}'}\Big[|0\rangle\langle 0|_R \otimes K_{\mathbf{P}}^{(A)}\rho_{\text{in}}K_{\mathbf{P}'}^{(A)\,\dagger} + |1\rangle\langle 1|_R \otimes K_{\mathbf{P}}^{(B)}\rho_{\text{in}}K_{\mathbf{P}'}^{(B)\,\dagger}\Big] + \text{off-diagonal terms in } R. \tag{335}$$

The *off-diagonal* block in the $\{|0\rangle_R, |1\rangle_R\}$ basis is (again up to an overall factor)

$$\rho_{\text{off}}^{(L)} = \frac{1}{2}\sum_{\mathbf{P},\mathbf{P}'} |0\rangle\langle 1|_R \otimes K_{\mathbf{P}}^{(A)}\, \rho_{\text{in}}\, K_{\mathbf{P}'}^{(B)\,\dagger}. \tag{336}$$

When we trace out the environments $\{E_x\}$, only the terms with matching classical flags $\mathbf{P} = \mathbf{P}'$ survive, because $\langle P_x | P_x'\rangle = \delta_{P_x, P_x'}$ for each $x \in Q_L$. Hence the off-diagonal block after tracing out the environments is

$$\rho_{\text{off,sys}}^{(L)} = \frac{1}{2}\sum_{\mathbf{P}} |0\rangle\langle 1|_R \otimes \Big(\prod_{x\in Q_L}\sqrt{q_x^{(A)}(P_x)\, q_x^{(B)}(P_x)}\Big)\Big(\bigotimes_{x\in Q_L} P_x\Big)\rho_{\text{in}}\Big(\bigotimes_{x\in Q_L} P_x\Big)^{\dagger}. \tag{337}$$

The holosymmetric process overlap $C^{(L)}$ is (up to a fixed normalization) the trace of this off-diagonal block against the corresponding ideal (noiseless) off-diagonal block; in the interaction picture the ideal block is just $|0\rangle\langle 1|_R \otimes \rho_{\text{in}}$, so

$$C^{(L)} = \sum_{\mathbf{P}}\Big(\prod_{x\in Q_L}\sqrt{q_x^{(A)}(P_x)\, q_x^{(B)}(P_x)}\Big)\, \mathrm{Tr}\Big[\rho_{\text{in}}\Big(\bigotimes_{x\in Q_L} P_x\Big)\rho_{\text{in}}\Big(\bigotimes_{x\in Q_L} P_x\Big)^{\dagger}\Big]. \tag{338}$$

For Pauli error channels acting symmetrically on the input (e.g. logical eigenstates or maximally mixed over the code), the trace factor in square brackets is independent of the specific Pauli pattern **P** and can be normalized to 1 (this is the usual situation in Pauli-channel process fidelity calculations [5]). Thus

$$C^{(L)} = \sum_{\mathbf{P}} \prod_{x \in Q_L} \sqrt{q_x^{(A)}(P_x)\, q_x^{(B)}(P_x)} = \prod_{x \in Q_L} \Bigg(\sum_{P \in \{\mathbb{I},X,Y,Z\}} \sqrt{q_x^{(A)}(P)\, q_x^{(B)}(P)} \Bigg) = \prod_{x \in Q_L} k_x, \quad (339)$$

where in the second equality we factor the sum over patterns into a product of single-site sums. This proves the factorization. In the homogeneous case $k_x \equiv k$ we obtain $C^{(L)} = k^{m_L}$, and substituting into $v_{\text{Holo}}^{(L)} = \frac{1}{2}(1 + C^{(L)})$ yields (331). $\square$

Remark 6.13. The single-site overlap k_x is precisely the classical Bhattacharyya (or Hellinger) overlap between the Pauli error distributions $\mathbf{q}_x^{(A)}$ and $\mathbf{q}_x^{(B)}$. Theorem 6.12 says that, for independent Pauli noise layers, the holosymmetric process overlap is the product of these classical overlaps across all touched PQS, mirroring standard multiplicativity of process fidelities for tensor-product Pauli channels [5, 8].

Definition 6.14 (Differential noise parameter)**.** In the homogeneous case of Theorem 6.12, define the *differential noise parameter*

$$\delta := 1 - k. \quad (340)$$

Then $k = 1 - \delta$ and

$$k^{m_L} = (1 - \delta)^{m_L}. \quad (341)$$

Corollary 6.15 (Bounds on $v_{\text{Holo}}^{(L)}$ in terms of δ)**.** *In the homogeneous Pauli case of Theorem Theorem 6.12, one has*

$$v_{\text{Holo}}^{(L)} = \frac{1}{2}\Big(1 + (1 - \delta)^{m_L}\Big). \quad (342)$$

In particular,

$$1 - v_{\text{Holo}}^{(L)} = \frac{1}{2}\Big(1 - (1 - \delta)^{m_L}\Big) \ \le\ \frac{1}{2}\, m_L\, \delta, \quad (343)$$

where we used $1 - (1 - \delta)^{m_L} \le m_L \delta$ *for* $\delta \in [0, 1]$.

Proof. The first identity is just (331) with $k = 1 - \delta$. For the inequality, note that for any $\delta \in [0, 1]$ and integer $m_L \ge 1$, the Bernoulli inequality gives

$$(1 - \delta)^{m_L} \ge 1 - m_L \delta,$$

which is equivalent to

$$1 - (1 - \delta)^{m_L} \le m_L \delta.$$

Substituting into $1 - v_{\text{Holo}}^{(L)} = \frac{1}{2}\Big(1 - (1 - \delta)^{m_L}\Big)$ yields

$$1 - v_{\text{Holo}}^{(L)} \le \frac{1}{2}\, m_L\, \delta,$$

as claimed. $\square$

Remark 6.16. Corollary 6.15 shows that, in the homogeneous independent-Pauli regime, the degradation of the holosymmetric valuation per cycle scales at most linearly in $m_L\delta$ for small differential noise δ. In later sections we will combine this with the distance scaling $m_L = O(L^2)$ for a fixed schedule and the logical threshold behaviour in Theorem 5.28 to obtain a quantitative *process-level* IQR threshold in terms of $(p, \delta, \varepsilon_{\mathrm{ps}})$ and the code distance d_L [1, 4, 5, 8].

Multi-cycle process overlap under stationarity

Assumption 6.17 (Stationarity and independence across cycles)**.** *Assume that each QEC cycle uses the same pair of schedules A, B and that the Pauli noise distributions $\mathbf{q}_x^{(A)}$, $\mathbf{q}_x^{(B)}$ are stationary from cycle to cycle and independent between cycles, as in standard Markovian local-noise assumptions in threshold theorems [2, 17, 3, 5].*

Definition 6.18 (Multi-cycle holosymmetric valuation)**.** Under Assumption 6.17, the dual-history comparison can be extended to T successive QEC cycles by coherently applying the controlled schedules T times, with the reference qubit R passed through all cycles and a final Hadamard H_R applied at the end, in line with coherent multi-round fault-tolerant gadgets [2, 3, 5]. The resulting holosymmetric valuation is denoted

$$v_{\mathrm{Holo}}^{(L,T)}.$$

We define the corresponding *multi-cycle holosymmetric process overlap* by

$$C^{(L,T)} := 2\, v_{\mathrm{Holo}}^{(L,T)} - 1, \tag{344}$$

so that $v_{\mathrm{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + C^{(L,T)}\Big)$, in direct analogy with the single-cycle case of Definition 6.4.

Lemma 6.19 (Multi-cycle process overlap for homogeneous Pauli noise)**.** *Under Assumptions 6.10 and 6.17, in the homogeneous case $k_x \equiv k$ one has*

$$C^{(L,T)} = k^{m_L T}, \tag{345}$$

and hence

$$v_{\mathrm{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + k^{m_L T}\Big) = \frac{1}{2}\Big(1 + (1-\delta)^{m_L T}\Big), \tag{346}$$

where $\delta = 1 - k$ as in Definition 6.14. Consequently,

$$1 - v_{\mathrm{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 - (1-\delta)^{m_L T}\Big) \;\le\; \frac{1}{2} m_L T\, \delta. \tag{347}$$

Proof. We first compute $C^{(L,T)}$ and then derive the bound on $1 - v_{\mathrm{Holo}}^{(L,T)}$.

Step 1: View T cycles as a single extended gadget.

Consider T successive QEC cycles with the same pair of schedules A, B at size L. Label the cycles by $t = 1, \ldots, T$. For each cycle t, let Q_L denote the set of PQS loci touched by either schedule within that cycle, with $|Q_L| = m_L$ as in Assumption 6.10.

Over all T cycles, the total collection of *noise events* can be labelled by pairs (x, t) with $x \in Q_L$ and $t \in \{1, \ldots, T\}$. Thus there are $m_L T$ single-qubit Pauli noise applications in each branch (A or B) across the full T-cycle history. By Assumption 6.10, these are independent across x within a cycle; by Assumption 6.17, the distributions are stationary and independent across t:

$$\mathbf{q}_{x,t}^{(S)} = \mathbf{q}_x^{(S)} \quad \text{for all } t,\ S \in \{A, B\},$$

and the joint error distribution for a fixed branch S across all (x, t) factors as a product of single-site single-cycle marginals.

We may therefore regard the entire T-cycle process as a *single* noisy multi-qubit gadget acting on the emergent register, with independent Pauli noise at each "spacetime site" (x, t).

Step 2: Apply the Pauli-channel holosymmetry factorization.

For a single cycle, Theorem 6.12 shows that, under independent Pauli noise on the set Q_L of size m_L, the holosymmetric process overlap is

$$C^{(L)} = \prod_{x \in Q_L} k_x, \tag{348}$$

where $k_x = \sum_P \sqrt{q_x^{(A)}(P) q_x^{(B)}(P)}$ is the single-site Pauli overlap.

By the same reasoning, applied now to the extended gadget with noise events labelled by (x, t), the holosymmetric overlap across *all* T cycles can be written as a product over all spacetime sites:

$$C^{(L,T)} = \prod_{t=1}^{T} \prod_{x \in Q_L} k_{x,t}, \tag{349}$$

where

$$k_{x,t} := \sum_{P \in \{\mathbb{I}, X, Y, Z\}} \sqrt{q_{x,t}^{(A)}(P)\, q_{x,t}^{(B)}(P)}.$$

Stationarity implies $q_{x,t}^{(S)}(P) = q_x^{(S)}(P)$ for all t, hence $k_{x,t} = k_x$ for all t. Therefore

$$C^{(L,T)} = \prod_{t=1}^{T} \prod_{x \in Q_L} k_x = \Big(\prod_{x \in Q_L} k_x\Big)^T = \Big(C^{(L)}\Big)^T. \tag{350}$$

In the homogeneous case $k_x \equiv k$ (Definition 6.11), we have $C^{(L)} = k^{m_L}$ by Theorem 6.12, and hence

$$C^{(L,T)} = \Big(k^{m_L}\Big)^T = k^{m_L T}. \tag{351}$$

Step 3: Expression for $v_{\text{Holo}}^{(L,T)}$.

By Definition 6.18, $v_{\text{Holo}}^{(L,T)}$ and $C^{(L,T)}$ are related by

$$v_{\text{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + C^{(L,T)}\Big),$$

so substituting $C^{(L,T)} = k^{m_L T}$ yields

$$v_{\text{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + k^{m_L T}\Big). \tag{352}$$

Using $\delta = 1 - k$ from Definition 6.14, we may also write $k^{m_L T} = (1-\delta)^{m_L T}$, giving

$$v_{\text{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + (1-\delta)^{m_L T}\Big). \tag{353}$$

Step 4: Upper bound on $1 - v_{\text{Holo}}^{(L,T)}$.

From the previous expression,

$$1 - v_{\text{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 - (1-\delta)^{m_L T}\Big).$$

For $\delta \in [0,1]$ and integer $N \geq 1$, Bernoulli's inequality gives

$$(1-\delta)^N \geq 1 - N\delta,$$

which is equivalent to

$$1 - (1-\delta)^N \leq N\delta.$$

Taking $N = m_L T$ we obtain

$$1 - (1-\delta)^{m_L T} \leq m_L T\,\delta,$$

and hence

$$1 - v_{\text{Holo}}^{(L,T)} \leq \frac{1}{2} m_L T\,\delta.$$

This completes the proof. □

6.3 Process Threshold Condition

We now formulate a process-level threshold condition ensuring that the process-equivalence proposition $\varphi_{\text{proc}}^{(L)}$ remains in the Thesis corner (holosymmetrically equivalent histories) over timescales of interest, in direct analogy with logical threshold conditions in standard fault-tolerant schemes [2, 17, 3, 5, 8].

Definition 6.20 (Process threshold function)**.** Let $\epsilon > 0$ be a target accuracy, and fix L and T. Define the *process threshold* $\delta_*(L,T;\epsilon)$ by

$$\delta_*(L,T;\epsilon) := \frac{2\epsilon}{m_L T}, \tag{354}$$

where m_L is the number of PQS touched per cycle (cf. the gate- or location-counting used in fault-tolerant threshold analyses [2, 3, 5]).

Theorem 6.21 (Process-equivalence threshold for holosymmetry)**.** *Let the assumptions of Lemma 6.19 hold (homogeneous Pauli noise, stationarity, and independence across cycles), and let $\delta := 1 - k$ be the differential noise parameter. Suppose that*

$$\delta \leq \delta_*(L,T;\epsilon) = \frac{2\epsilon}{m_L T}. \tag{355}$$

Then the multi-cycle holosymmetric valuation satisfies

$$v_{\text{Holo}}^{(L,T)} \geq 1 - \epsilon. \tag{356}$$

Equivalently, the deficit of the process Thesis valuation obeys

$$1 - v_{\text{Holo}}^{(L,T)} \leq \epsilon. \tag{357}$$

Proof. By Lemma 6.19,

$$1 - v_{\text{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 - (1-\delta)^{m_L T}\Big) \;\leq\; \frac{1}{2} m_L T\,\delta. \tag{358}$$

If $\delta \leq 2\epsilon/(m_L T)$, then

$$1 - v_{\text{Holo}}^{(L,T)} \leq \frac{1}{2} m_L T \cdot \frac{2\epsilon}{m_L T} = \epsilon,$$

as claimed. □

Corollary 6.22 (Process threshold for polynomial timescales)**.** *Let d_L denote the distance of the surface code at size L, and let $T_L := \mathrm{poly}(d_L)$ be any fixed polynomial in d_L. Suppose there exists a constant $\kappa > 0$ such that $m_L \leq \kappa n_L$ and $n_L = \Theta(L^2)$, so in particular $m_L = O(L^2)$ and $d_L \sim L$ for the standard surface-code family [1, 4, 5]. Then for any $\epsilon > 0$, if*

$$\delta \;\leq\; \delta_*(L, T_L; \epsilon) = \frac{2\epsilon}{m_L T_L} = O\Big(\frac{\epsilon}{L^2 T_L}\Big), \tag{359}$$

the multi-cycle holosymmetric valuation satisfies

$$v_{\text{Holo}}^{(L,T_L)} \geq 1 - \epsilon. \tag{360}$$

In particular, for a fixed differential noise parameter $\delta > 0$, the timescale T_L over which process-equivalence remains in the Thesis corner ($v_{\text{Holo}}^{(L,T_L)} \geq 1 - \epsilon$) satisfies

$$T_L \;\leq\; \frac{2\epsilon}{\delta m_L} = O\Big(\frac{\epsilon}{\delta\, m_L}\Big), \tag{361}$$

mirroring the way logical lifetimes scale with distance and noise parameters in fault-tolerant threshold theorems [2, 17, 3, 5].

Remark 6.23. Corollary 6.22 shows that, in addition to the logical state-level threshold (Corollary 5.29), there exists a *process-level* threshold: if the differential noise parameter δ (which measures how similar the Pauli error distributions are for schedules A and B) is sufficiently small on the scale $1/(m_L T_L)$, then the holosymmetric Thesis valuation $v_{\text{Holo}}^{(L,T_L)}$ remains close to 1 for times T_L up to polynomial in the code distance, in line with standard expectations for robust fault-tolerant implementations of logical operations [2, 3, 5, 8].

In Tetralemma terms, this means that the process-equivalence proposition $\varphi_{\text{proc}}^{(L)}$ stays in the Thesis corner with valuation at least $1 - \epsilon$ over the timescales relevant for fault-tolerant computation, complementing the logical correctness control obtained in Phase 3 [1, 4, 5, 8].

7 Phase 5: The Full IQR Fault-Tolerance Threshold Theorem

In this section we bundle Phases 1–4 into a single, global threshold statement in the IQR / Tetralemma language. We state an IQR fault-tolerance threshold theorem for a surface-code family on the emergent Twofold register, and give a structured proof sketch referencing the results established in the previous phases. Conceptually, this plays the role of a threshold theorem in the IQR/Tetralemma setting, analogous to standard results in the stabilizer and topological-code literature [18, 19, 20, 13, 17, 2, 3, 1, 11, 12, 22, 4, 25, 14, 15, 5, 8].

7.1 The IQR Fault-Tolerance Threshold Theorem

We first collect the standing assumptions from Phases 1–4 in a compact form.

Assumption 7.1 (IQR surface-code setting and noise hypotheses)**.** *We assume:*

(i) Ambient IQR and emergent register.
The ambient IQR data of Assumption 2.1 holds on a region $\mathcal{U} \subset \mathcal{X}_q$*, and for each* $L \in \mathbb{N}$ *there is a patch* $\mathcal{P}_L \subset \mathcal{U}$ *with emergent Twofold register*

$$\mathcal{H}_{\text{reg}}^{(L)} = \bigotimes_{x_j \in \mathcal{P}_L} \mathcal{H}_{(0,1),x_j}$$

as in Definition 5.2, arising from the IQR correlator and Holo-State construction [8].

(ii) Surface-code family.
On each $\mathcal{H}_{\text{reg}}^{(L)}$ *there is a surface-code patch* $\mathcal{C}_L \subset \mathcal{H}_{\text{reg}}^{(L)}$ *encoding one logical qubit, with stabilizer group* S_L*, projector* $P_{\text{SC},L} = |S_L|^{-1} \sum_{g \in S_L} g$*, and distance* $d_L \sim L$ *as in Definition 5.3. This realizes a standard 2D topological stabilizer-code family with local checks and growing distance [16, 21, 1, 11, 12, 14, 15, 4, 5]. The associated logical correctness proposition* $\varphi_L^{(L)}$ *and Thesis valuation* $v_{\text{Th}}(\rho_{\text{out}}; \varphi_L^{(L)})$ *are defined as in Definitions 2.6 and 2.9.*

(iii) Fourfold correlator and IQR noise class.
The actual Fourfold correlator $\hat{G}_\Lambda$ *and the ideal correlator* $\hat{G}_\Lambda^0$ *define a noise kernel* $\Delta\hat{G} = \hat{G}_\Lambda - \hat{G}_\Lambda^0$ *(Definition 4.1) that lies in the IQR noise class of Assumption 4.5: finite spatial correlation length* ξ*, finite correlation time* t_{corr}*, and smallness in the Twofold block* $\Delta\hat{G}_{(0,1)}$ *with bound* ϵ_0*, in line with cluster-type assumptions used in rigorous local-noise analyses [2, 3, 5, 8].*

(iv) Small Twofold–(p, s) couplings (leakage).
The Kraus operators of one QEC cycle on the full Hilbert space admit a block structure as in Assumption 3.17, with off-block operator norms uniformly bounded by η_{ps}*. The induced leakage parameter* ε_{ps} *defined in Definition 3.15 is small and stable under composition (Lemmas 3.18 and 3.19), reflecting the controlled-leakage regime considered in surface-code and fault-tolerance studies [25, 5].*

(v) Effective local Pauli noise.
Under the IQR noise-class conditions and approximate Pauli covariance (Assumption 4.13), the effective noise on the emergent register per QEC cycle is, up to small correlated and leakage corrections, a tensor product of single-qubit Pauli channels with bounded error rate p:

$$\mathcal{N}_{\text{eff}}^{(n_L)} = \left(\bigotimes_{x_j \in \mathcal{P}_L} \Lambda_j \right) + \mathcal{N}_{\text{corr}}^{(n_L)} + \mathcal{N}_{\text{leak}}^{(n_L)}, \tag{362}$$

where Λ_j *acts on* $\mathcal{H}_{(0,1),x_j}$ *with local Pauli error rate* $p^{(j)} \le p$*, and* $\|\mathcal{N}_{\text{corr}}^{(n_L)}\|_\diamond \le \epsilon_{\text{corr}}(n_L)$*,* $\|\mathcal{N}_{\text{leak}}^{(n_L)}\|_\diamond = O(\varepsilon_{\text{ps}})$ *as in Corollary 4.20. This places the emergent register in the standard local Pauli-noise regime underpinning surface-code threshold theorems [18, 20, 13, 17, 2, 3, 1, 22, 4, 5].*

(vi) Differential noise between two QEC schedules.

For each L there exist two full QEC schedules A and B implementing the same ideal logical channel $\mathcal{N}_{\mathrm{ideal}}^{(L)}$ on $\mathcal{C}_L$, with independent single-qubit Pauli noise layers characterized by per-site distributions $\mathbf{q}_x^{(A)}$, $\mathbf{q}_x^{(B)}$ (Assumption 6.10), and single-site Pauli overlap

$$k_x := \sum_{P \in \{\mathbb{I}, X, Y, Z\}} \sqrt{q_x^{(A)}(P)\, q_x^{(B)}(P)}.$$

In the homogeneous case we write $k_x \equiv k$ for x in the set Q_L of PQS touched by the gadgets, $|Q_L| = m_L$. The differential noise parameter is

$$\delta := 1 - k. \tag{363}$$

We define a process-equivalence proposition $\varphi_{\mathrm{proc}}^{(L)}$ and holosymmetric Thesis valuation $v_{\mathrm{Holo}}^{(L,T)}$ as in Definitions 2.16 and 6.18, with the multi-cycle process overlap given by Lemma 6.19. This mirrors the comparison of alternative fault-tolerant gadgets and schedules in circuit-based constructions [2, 17, 3, 5].

We can now state the IQR fault-tolerance threshold theorem.

Theorem 7.2 (IQR fault-tolerance threshold, surface-code version)**.** *Assume the IQR surface-code setting and noise hypotheses of Assumption 7.1. Then there exist:*

- *a* logical noise threshold *$p_* > 0$,*
- *constants $c > 0$ and $C > 0$ (independent of L),*
- *a* leakage scale *constant $C_{\mathrm{leak}} > 0$ such that $\|\mathcal{N}_{\mathrm{leak}}^{(n_L)}\|_\diamond \le C_{\mathrm{leak}} \varepsilon_{\mathrm{ps}}$,*
- *and, for each code size L and time horizon T, a* process threshold *$\delta_*(L,T) > 0$,*

such that the following holds.

For any target accuracy $\epsilon > 0$, there exists a code size L and a time horizon $T = \mathrm{poly}(d_L)$ (polynomial in the code distance) such that, if

$$p < p_*, \qquad C_{\mathrm{leak}} \varepsilon_{\mathrm{ps}} \ll 1, \qquad \epsilon_{\mathrm{corr}}(n_L) \ll 1, \qquad \delta < \delta_*(L,T), \tag{364}$$

then:

(i) Logical correctness.

The Thesis valuation of the logical correctness proposition $\varphi_L^{(L)}$ satisfies

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) \;=\; 1 - p_L^{(L)} \;\ge\; 1 - C\Big(\frac{p}{p_*}\Big)^{c\, d_L} - O\Big(\epsilon_{\mathrm{corr}}(n_L)\Big) - O\Big(\varepsilon_{\mathrm{ps}}\Big), \tag{365}$$

and, for L sufficiently large and $(p, \epsilon_{\mathrm{corr}}(n_L), \varepsilon_{\mathrm{ps}})$ in the threshold-compatible regime,

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) > 1 - \epsilon. \tag{366}$$

Equivalently, the worst-case logical error probability per cycle satisfies

$$p_L^{(L)} \;\leq\; C\Big(\frac{p}{p_*}\Big)^{c\,d_L} + O\big(\epsilon_{\mathrm{corr}}(n_L)\big) + O\big(\varepsilon_{\mathrm{ps}}\big) \;<\; \epsilon, \tag{367}$$

where $p_L^{(L)}$ *is defined in Definition 2.7. The existence of such a threshold* p_* *and exponential-in-distance decay of* $p_L^{(L)}$ *under local Pauli noise are in direct correspondence with standard surface-code threshold results and their refinements under weak correlations and leakage [1, 11, 12, 22, 4, 25, 15, 5, 23, 24].*

(ii) Process equivalence over T QEC cycles.

The multi-cycle holosymmetric Thesis valuation for the process-equivalence proposition $\varphi^{(L)}_{\mathrm{proc}}$ *satisfies*

$$v_{\mathrm{Holo}}^{(L,T)} \;\geq\; 1-\epsilon. \tag{368}$$

In particular, in the homogeneous Pauli case, Lemma 6.19 gives

$$v_{\mathrm{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + (1-\delta)^{m_L T}\Big), \tag{369}$$

and the process threshold can be chosen as

$$\delta_*(L,T) := \frac{2\epsilon}{m_L T}, \tag{370}$$

ensuring $1 - v_{\mathrm{Holo}}^{(L,T)} \leq \epsilon$*, in analogy with schedule- and gadget-level robustness conditions in circuit-based fault-tolerance [2, 17, 3, 5].*

Consequently, by choosing L *sufficiently large and taking* $(p, \delta, \varepsilon_{\mathrm{ps}})$ *within the threshold regime, both the state-level logical proposition* $\varphi_L^{(L)}$ *and the process-equivalence proposition* $\varphi^{(L)}_{\mathrm{proc}}$ *reside stably in the* Thesis *corner of the Tetralemma, with the combined weight of the AntiThesis, Synthesis, and Holothesis valuations bounded by* ϵ*, thereby providing an IQR-based analogue of fault-tolerant quantum computation in the standard framework [18, 13, 17, 2, 3, 1, 4, 5, 8].*

7.1.1 The IQR Fault-Tolerance Threshold Theorem Proof Roadmap

We now provide a structured proof sketch for Theorem 7.2, indicating how each phase contributes to the final result.

Proof sketch. We proceed in four conceptual steps corresponding to Phases 1–4.

Step 1: Emergent Twofold register and leakage control (Phase 1).

From Assumptions 2.1 and 3.3, we construct the modal fibres $\mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}}$ and their orthonormal Fourfold bases $\{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$ via the coincident-limit inner product (Lemma 3.7), in the IQR framework of [8]. This yields a smooth decomposition

$$\mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}} = \mathcal{H}_{(0,1),x} \oplus \mathcal{H}_{(p,s),x},$$

with $\mathcal{H}_{(0,1),x}$ the emergent qubit at x (Lemma 3.9). For a patch $\mathcal{P}_L$, we obtain the emergent Twofold register

$$\mathcal{H}_{\mathrm{reg}}^{(L)} = \bigotimes_{x_j \in \mathcal{P}_L} \mathcal{H}_{(0,1),x_j},$$

which will host the surface-code family [13, 21, 1, 14, 15].

By introducing the global projector P_{reg} onto $\mathcal{H}_{\mathrm{reg}}^{(L)}$ and its complement P_{comp}, we define the confined register channel $\mathcal{N}_{\mathrm{reg}}$ and the leakage parameter

$$\varepsilon_{\mathrm{ps}} := \sup_{\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{reg}}^{(L)})} \left\| \mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}_{\mathrm{reg}}(\rho) \right\|_1,$$

as in Definition 3.15. Under the block-Kraus Assumption 3.17, Lemma 3.18 bounds $\varepsilon_{\mathrm{ps}}$ in terms of off-block norms, and Lemma 3.19 shows that leakage grows at most linearly in the number of QEC cycles, in line with standard treatments of leakage in fault-tolerant schemes [17, 3, 5].

Step 2: From IQR correlators to effective local Pauli noise (Phase 2).

Under the IQR noise-class conditions on $\Delta\hat{G}$ (Assumption 4.5), the Twofold block deviations $\Delta\hat{G}_{(0,1)}$ are uniformly small. These deviations control the difference between the actual and ideal local channels at each site via their Choi matrices (Lemma 4.10), leading to a diamond-norm bound

$$\left\| \mathcal{N}_j^{(1)} - \mathcal{N}_j^{(1),0} \right\|_\diamond \le K' \epsilon_0$$

(Lemma 4.11). With approximate Pauli covariance (Assumption 4.13), we introduce associated Pauli channels Λ_j and show that each local Twofold channel is close to Λ_j in diamond norm (Lemma 4.15), with

$$\left\| \mathcal{N}_j^{(1)} - \Lambda_j \right\|_\diamond \le \epsilon_{\mathrm{loc}} = O(\epsilon_0 + \epsilon_{\mathrm{twirl}}),$$

and local error rate $p^{(j)} \le p$, paralleling local-noise models used in rigorous threshold analyses [2, 3, 16].

Exponential clustering of $\Delta\hat{G}$ implies that multi-site correlations are exponentially suppressed in the cluster diameter (Lemmas 4.17 and 4.18), so the global register channel $\mathcal{N}^{(n_L)}$ decomposes as

$$\mathcal{N}^{(n_L)} = \Lambda^{(n_L)} + \mathcal{N}_{\mathrm{corr}}^{(n_L)}, \qquad \Lambda^{(n_L)} := \bigotimes_{x_j \in \mathcal{P}_L} \Lambda_j,$$

with

$$\left\| \mathcal{N}_{\mathrm{corr}}^{(n_L)} \right\|_\diamond \le \epsilon_{\mathrm{corr}}(n_L)$$

(Lemma 4.18). Together with leakage control from Step 1, this yields the effective Pauli noise regime summarized in Corollary 4.20 and Equation (362), placing the emergent register in the standard "local Pauli noise plus small corrections" setting used throughout the fault-tolerance literature [18, 19, 20, 13, 17, 2, 3].

Step 3: Surface-code logical threshold (Phase 3).

Given the effective noise model on $\mathcal{H}_{\mathrm{reg}}^{(L)}$, Assumption 5.27 (small p, $\epsilon_{\mathrm{corr}}(n_L)$, and $\varepsilon_{\mathrm{ps}}$) places us in the regime of surface-code threshold theorems for 2D topological stabilizer codes [1, 11, 12, 22, 4, 25, 14, 15, 5]. By Theorem 5.28, there exist $p_* > 0$, $c > 0$, and $C > 0$ such that for $p < p_*$ the worst-case logical error probability satisfies

$$p_L^{(L)} \le C\Big(\frac{p}{p_*}\Big)^{c\,d_L} + O\big(\epsilon_{\mathrm{corr}}(n_L)\big) + O\big(\varepsilon_{\mathrm{ps}}\big),$$

where the leading exponential decay in d_L reflects the standard threshold behaviour of surface codes with efficient decoders (e.g. [1, 4, 5, 23, 24]). Translating into the Thesis valuation $v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) = 1 - p_L^{(L)}$ yields Corollary 5.29,

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) \ge 1 - C\Big(\frac{p}{p_*}\Big)^{c\,d_L} - O\big(\epsilon_{\mathrm{corr}}(n_L)\big) - O\big(\varepsilon_{\mathrm{ps}}\big).$$

By choosing L sufficiently large and ensuring that $\epsilon_{\mathrm{corr}}(n_L)$ and $\varepsilon_{\mathrm{ps}}$ are sufficiently small, we can make the logical Thesis valuation exceed $1 - \epsilon/3$, say.

Step 4: Holosymmetric process threshold (Phase 4).

For the process layer, we consider two QEC schedules A and B and the dual-history interferometer of Definition 6.3. In the homogeneous Pauli case, Theorem 6.12 shows that the single-cycle process overlap is

$$C^{(L)} = k^{m_L},$$

where m_L is the number of PQS touched per cycle and $k = 1 - \delta$ is the single-site Pauli overlap (Definition 6.14). Thus

$$v_{\mathrm{Holo}}^{(L)} = \frac{1}{2}\Big(1 + (1-\delta)^{m_L}\Big),$$

and

$$1 - v_{\mathrm{Holo}}^{(L)} = \frac{1}{2}\Big(1 - (1-\delta)^{m_L}\Big) \le \frac{1}{2} m_L \delta$$

(Corollary 6.15).

Under stationarity and independence across cycles (Assumption 6.17), Lemma 6.19 gives

$$v_{\mathrm{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + (1-\delta)^{m_L T}\Big),$$

so

$$1 - v_{\mathrm{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 - (1-\delta)^{m_L T}\Big) \le \frac{1}{2} m_L T\,\delta.$$

The process threshold Theorem 6.21 then states that if

$$\delta \le \delta_*(L,T) := \frac{2\epsilon}{m_L T},$$

we obtain $1 - v_{\mathrm{Holo}}^{(L,T)} \le \epsilon$. In particular, for a given polynomial time horizon $T = \mathrm{poly}(d_L)$, choosing δ sufficiently small (of order $1/(m_L T)$) ensures that the process-equivalence proposition $\varphi_{\mathrm{proc}}^{(L)}$ remains in the Thesis corner with valuation at least $1 - \epsilon/3$, say.

Step 5: Combining logical and process thresholds.

Fix a target accuracy $\epsilon > 0$. Choose small fractions $\epsilon_{\mathrm{log}}, \epsilon_{\mathrm{proc}}, \epsilon_{\mathrm{leak}} > 0$ with

$$\epsilon_{\mathrm{log}} + \epsilon_{\mathrm{proc}} + \epsilon_{\mathrm{leak}} \le \epsilon.$$

- *Logical part:* Choose L sufficiently large and $p < p_*$ such that

  $$C\left(\frac{p}{p_*}\right)^{c\,d_L} \le \epsilon_{\mathrm{log}},$$

  and ensure $\epsilon_{\mathrm{corr}}(n_L)$ and $\varepsilon_{\mathrm{ps}}$ are small enough that the additional terms in Theorem 5.28 are $\le \epsilon_{\mathrm{leak}}$. Then

  $$1 - v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) \le \epsilon_{\mathrm{log}} + \epsilon_{\mathrm{leak}} \le \epsilon.$$

- *Process part:* For a chosen polynomial time horizon $T = \mathrm{poly}(d_L)$, set

  $$\delta_*(L,T) := \frac{2\epsilon_{\mathrm{proc}}}{m_L T}$$

  and demand that the differential noise parameter satisfies $\delta < \delta_*(L,T)$. Then, by Theorem 6.21,

  $$1 - v_{\mathrm{Holo}}^{(L,T)} \le \epsilon_{\mathrm{proc}} \le \epsilon.$$

Thus, for suitable choices of L, T, and noise parameters $(p, \delta, \varepsilon_{\mathrm{ps}})$ within the threshold regime, both the logical correctness proposition $\varphi_L^{(L)}$ and the process-equivalence proposition $\varphi_{\mathrm{proc}}^{(L)}$ have Thesis valuations at least $1 - \epsilon$.

In Tetralemma semantics (as developed in [8]), this means that the total weight assigned to the AntiThesis, Synthesis, and Holothesis corners for these propositions is at most ϵ. This completes the proof sketch of Theorem 7.2. □

7.1.2 Proof of the IQR Fault-Tolerance Threshold Theorem

We now give a detailed proof of Theorem 7.2, making the dependence on the constants and parameters explicit and indicating precisely where the results of Phases 1–4 enter.

Proof. Fix a target accuracy $\epsilon > 0$. We will exhibit:

- a logical noise threshold $p_* > 0$ and constants $c, C > 0$;
- a leakage threshold $\varepsilon_{\mathrm{ps},*} > 0$;
- and, for each code size L and time horizon T, a process threshold $\delta_*(L,T) > 0$;

such that the conclusions (i) and (ii) of Theorem 7.2 hold whenever

$$p < p_*, \qquad \varepsilon_{\mathrm{ps}} < \varepsilon_{\mathrm{ps},*}, \qquad \delta < \delta_*(L,T),$$

for suitable L and $T = \mathrm{poly}(d_L)$.

We proceed in four steps, reflecting Phases 1–4 of the construction.

Step 1: Emergent register and leakage control (Phase 1).

By Assumptions 2.1 and 3.3, the IQR framework together with the coincident-limit inner product yields, for each $x \in \mathcal{U}$, a modal fibre $\mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)}$ and an orthonormal Fourfold basis

$\{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$ as constructed in Lemma 3.7 (cf. [8]). This gives a smooth orthogonal decomposition

$$\mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)} = \mathcal{H}_{(0,1),x} \oplus \mathcal{H}_{(p,s),x}, \tag{371}$$

with $\mathcal{H}_{(0,1),x}$ the emergent qubit at x, varying smoothly across $\mathcal{U}$ (Lemma 3.9).

For a surface-code patch $\mathcal{P}_L \subset \mathcal{U}$ (Definition 5.1), the emergent Twofold register is (Definition 5.2)

$$\mathcal{H}_{\mathrm{reg}}^{(L)} := \bigotimes_{x_j \in \mathcal{P}_L} \mathcal{H}_{(0,1),x_j}. \tag{372}$$

Let P_{reg} denote the projector onto $\mathcal{H}_{\mathrm{reg}}^{(L)}$ inside the full GNS Hilbert space $\mathcal{H}_{\mathrm{em}}$, and $P_{\mathrm{comp}} := \mathbb{I} - P_{\mathrm{reg}}$ the complementary projector.

Let $\mathcal{N}$ be the CPTP map describing one full physical QEC cycle on $\mathcal{H}_{\mathrm{em}}$. We define the confined register channel $\mathcal{N}_{\mathrm{reg}}$ by sandwiching $\mathcal{N}$ between P_{reg} and tracing out the complementary sector, as in Definition 3.14. The leakage parameter is then (Definition 3.15)

$$\varepsilon_{\mathrm{ps}} := \sup_{\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{reg}}^{(L)})} \left\| \mathrm{Tr}_{\mathrm{comp}}(\mathcal{N}(\rho)) - \mathcal{N}_{\mathrm{reg}}(\rho) \right\|_1. \tag{373}$$

Under the block-Kraus Assumption 3.17, Lemma 3.18 implies that there exists a constant $C_{\mathrm{ps}} > 0$, independent of L, such that

$$\varepsilon_{\mathrm{ps}} \le C_{\mathrm{ps}} \max_{\alpha} \left(\left\| P_{\mathrm{comp}} K_\alpha P_{\mathrm{reg}} \right\| + \left\| P_{\mathrm{reg}} K_\alpha P_{\mathrm{comp}} \right\| \right), \tag{374}$$

where $\{K_\alpha\}_\alpha$ is any Kraus family for $\mathcal{N}$. In particular, small off-block Kraus norms imply small leakage.

Moreover, Lemma 3.19 shows that if we consider T successive cycles, each described by a channel whose leakage parameter is bounded by the same $\varepsilon_{\mathrm{ps}}$, then the total leakage accumulated over T cycles is at most linear in T:

$$\varepsilon_{\mathrm{ps}}^{(T)} \le T\, \varepsilon_{\mathrm{ps}}. \tag{375}$$

This is completely analogous to leakage accumulation bounds in standard fault-tolerant schemes [17, 3, 5].

Step 2: Effective local Pauli noise from the Fourfold correlator (Phase 2).

Now invoke the IQR noise-class Assumption 4.5 on the Fourfold noise kernel $\Delta\hat{G} = \hat{G}_\Lambda - \hat{G}_\Lambda^0$ (Definition 4.1). In particular, the Twofold block $\Delta\hat{G}_{(0,1)}$ is uniformly small on $\mathcal{P}_L$, with bound ϵ_0, and exhibits finite correlation length ξ and correlation time t_{corr}.

For each site $x_j \in \mathcal{P}_L$, consider the local Twofold channel $\mathcal{N}_j^{(1)}$ and its ideal counterpart $\mathcal{N}_j^{(1),0}$ constructed from the ideal correlator $\hat{G}_{(0,1)}^0$ (Definition 4.7). By Lemma 4.10,

$$\left\| J(\mathcal{N}_j^{(1)}) - J(\mathcal{N}_j^{(1),0}) \right\|_1 \le K\, \epsilon_0 \tag{376}$$

for a constant $K > 0$ independent of j, where $J(\cdot)$ denotes the Choi matrix. Lemma 4.11 then gives a constant $K' > 0$ such that

$$\left\| \mathcal{N}_j^{(1)} - \mathcal{N}_j^{(1),0} \right\|_\diamond \le K'\, \epsilon_0. \tag{377}$$

Next, impose approximate Pauli covariance (Assumption 4.13). For each x_j, there exists a single-qubit Pauli channel Λ_j on $\mathcal{H}_{(0,1),x_j}$ such that (Lemma 4.15)

$$\left\| \mathcal{N}_j^{(1)} - \Lambda_j \right\|_\diamond \;\le\; \epsilon_{\text{loc}}, \qquad \epsilon_{\text{loc}} = O(\epsilon_0 + \epsilon_{\text{twirl}}), \tag{378}$$

and the associated local Pauli error rate $p^{(j)} := 1 - q_{\mathbb{I}}^{(j)}$ satisfies $p^{(j)} \le p$ for some uniform $p > 0$. This reproduces the standard local-Pauli-noise paradigm used in threshold analyses [2, 3, 16, 5].

To extend this to the full register, use the spatial clustering part of Assumption 4.5. Lemma 4.17 shows that the global register channel $\mathcal{N}^{(n_L)}$ admits a Kraus representation in which multi-site Kraus terms are exponentially suppressed in their spatial diameter with rate set by ξ. Lemma 4.18 then constructs a decomposition

$$\mathcal{N}^{(n_L)} \;=\; \Lambda^{(n_L)} + \mathcal{N}_{\text{corr}}^{(n_L)}, \qquad \Lambda^{(n_L)} := \bigotimes_{x_j \in \mathcal{P}_L} \Lambda_j, \tag{379}$$

with

$$\left\| \mathcal{N}_{\text{corr}}^{(n_L)} \right\|_\diamond \;\le\; \epsilon_{\text{corr}}(n_L), \qquad \epsilon_{\text{corr}}(n_L) = O\Big(n_L e^{-r/\xi}\Big), \tag{380}$$

for a suitable separation scale r.

Combining this with the leakage control of Step 1 yields precisely the decomposition of Corollary 4.20:

$$\mathcal{N}_{\text{eff}}^{(n_L)} = \Lambda^{(n_L)} + \mathcal{N}_{\text{corr}}^{(n_L)} + \mathcal{N}_{\text{leak}}^{(n_L)}, \tag{381}$$

with $\Lambda^{(n_L)}$ a tensor product of single-qubit Pauli channels with uniform error rate bound p, and

$$\left\| \mathcal{N}_{\text{corr}}^{(n_L)} \right\|_\diamond \;\le\; \epsilon_{\text{corr}}(n_L), \qquad \left\| \mathcal{N}_{\text{leak}}^{(n_L)} \right\|_\diamond \;=\; O(\varepsilon_{\text{ps}}), \tag{382}$$

placing the emergent register in the standard "local Pauli noise plus weak correlations and leakage" regime used throughout fault-tolerance theory [18, 20, 13, 17, 2, 3, 1, 22, 4, 5].

Step 3: Logical surface-code threshold and logical Thesis valuation (Phase 3). Assumption 7.1(ii) specifies a family of surface codes $\{\mathcal{C}_L\}$ on $\mathcal{H}_{\text{reg}}^{(L)}$ with distance $d_L \sim L$ (Definition 5.3). Under the noise model (381) and the threshold-compatible conditions of Assumption 5.27, the hypotheses of Theorem 5.28 are satisfied. Thus there exist constants $p_* > 0$, $c > 0$, and $C_0 > 0$, independent of L, such that for all $p < p_*$ the worst-case logical error probability per cycle obeys

$$p_L^{(L)} \;\le\; C_0 \Big(\frac{p}{p_*}\Big)^{c\,d_L} + C_1\, \epsilon_{\text{corr}}(n_L) + C_2\, \varepsilon_{\text{ps}}, \tag{383}$$

for some constants $C_1, C_2 > 0$ (using the $O(\cdot)$-notation from Theorem 5.28 to infer the existence of such constants).

By Definition 2.9,

$$v_{\text{Th}}(\rho_{\text{out}}; \varphi_L^{(L)}) \;=\; 1 - p_L^{(L)}, \tag{384}$$

so (383) implies

$$v_{\text{Th}}(\rho_{\text{out}}; \varphi_L^{(L)}) \;\ge\; 1 - C_0 \Big(\frac{p}{p_*}\Big)^{c\,d_L} - C_1\, \epsilon_{\text{corr}}(n_L) - C_2\, \varepsilon_{\text{ps}}. \tag{385}$$

This is precisely the logical Thesis-valuation bound of Corollary 5.29.

Now fix a decomposition of the accuracy budget into three positive parts,

$$\epsilon_{\mathrm{log}},\ \epsilon_{\mathrm{corr}},\ \epsilon_{\mathrm{leak}} > 0, \qquad \epsilon_{\mathrm{log}} + \epsilon_{\mathrm{corr}} + \epsilon_{\mathrm{leak}} \ \le\ \epsilon.$$

- Since $d_L \to \infty$ as $L \to \infty$ and $p/p_* < 1$, there exists $L_0 \in \mathbb{N}$ such that for all $L \ge L_0$,
$$C_0\Big(\frac{p}{p_*}\Big)^{c\,d_L} \ \le\ \epsilon_{\mathrm{log}}. \tag{386}$$
- By Assumption 5.27, we can choose the geometry and patch size so that $\epsilon_{\mathrm{corr}}(n_L)$ remains bounded or decays with L. Thus there exists $L_1 \in \mathbb{N}$ such that for all $L \ge L_1$,
$$C_1\,\epsilon_{\mathrm{corr}}(n_L) \ \le\ \epsilon_{\mathrm{corr}}. \tag{387}$$
- Choose a leakage threshold
$$\varepsilon_{\mathrm{ps},*} := \frac{\epsilon_{\mathrm{leak}}}{2C_2}, \tag{388}$$
and impose $\varepsilon_{\mathrm{ps}} < \varepsilon_{\mathrm{ps},*}$. Then
$$C_2\,\varepsilon_{\mathrm{ps}} \ \le\ \frac{\epsilon_{\mathrm{leak}}}{2} \ \le\ \epsilon_{\mathrm{leak}}. \tag{389}$$

Let $L_* := \max\{L_0, L_1\}$, and take any $L \ge L_*$. Substituting the preceding bounds into (385) yields

$$1 - v_{\mathrm{Th}}(\rho_{\mathrm{out}};\varphi_L^{(L)}) \ \le\ \epsilon_{\mathrm{log}} + \epsilon_{\mathrm{corr}} + \epsilon_{\mathrm{leak}} \ \le\ \epsilon. \tag{390}$$

Equivalently,

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}};\varphi_L^{(L)}) \ \ge\ 1 - \epsilon. \tag{391}$$

This proves the logical part (i) of Theorem 7.2.

Step 4: Holosymmetric process threshold and multi-cycle behaviour (Phase 4).

We now turn to the process layer. For each L, Assumption 7.1(vi) provides two QEC schedules A and B implementing the same ideal logical channel $\mathcal{N}_{\mathrm{ideal}}^{(L)}$ on $\mathcal{C}_L$, with per-site Pauli error distributions $\mathbf{q}_x^{(A)}$ and $\mathbf{q}_x^{(B)}$, and single-site overlaps

$$k_x = \sum_{P\in\{\mathbb{I},X,Y,Z\}} \sqrt{q_x^{(A)}(P)\,q_x^{(B)}(P)}. \tag{392}$$

Let Q_L be the set of PQS touched by these gadgets in one cycle, and $m_L := |Q_L|$. In the homogeneous case we set $k_x \equiv k$ for $x \in Q_L$ and define the differential noise parameter

$$\delta := 1 - k. \tag{393}$$

For a single cycle, Theorem 6.12 shows that the holosymmetric process overlap is

$$C^{(L)} = k^{m_L}, \tag{394}$$

and hence the holosymmetric valuation is

$$v_{\mathrm{Holo}}^{(L)} = \frac{1}{2}\Big(1 + k^{m_L}\Big) = \frac{1}{2}\Big(1 + (1-\delta)^{m_L}\Big). \tag{395}$$

Corollary 6.15 then gives the bound

$$1 - v_{\mathrm{Holo}}^{(L)} = \frac{1}{2}\Big(1 - (1-\delta)^{m_L}\Big) \;\le\; \frac{1}{2} m_L \delta. \tag{396}$$

Under stationarity and independence of the Pauli noise across cycles (Assumption 6.17), Lemma 6.19 extends this to T cycles:

$$C^{(L,T)} = k^{m_L T}, \qquad v_{\mathrm{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + (1-\delta)^{m_L T}\Big), \tag{397}$$

and

$$1 - v_{\mathrm{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 - (1-\delta)^{m_L T}\Big) \;\le\; \frac{1}{2} m_L T\,\delta. \tag{398}$$

Define the process threshold function as in Definition 6.20:

$$\delta_*(L,T;\epsilon_{\mathrm{proc}}) := \frac{2\epsilon_{\mathrm{proc}}}{m_L T}, \tag{399}$$

for any chosen $\epsilon_{\mathrm{proc}} > 0$. If $\delta \le \delta_*(L,T;\epsilon_{\mathrm{proc}})$, then Theorem 6.21 implies

$$1 - v_{\mathrm{Holo}}^{(L,T)} \;\le\; \epsilon_{\mathrm{proc}}. \tag{400}$$

Equivalently,

$$v_{\mathrm{Holo}}^{(L,T)} \;\ge\; 1 - \epsilon_{\mathrm{proc}}. \tag{401}$$

Now fix a polynomial timescale $T_L := \mathrm{poly}(d_L)$ (for example $T_L = d_L^k$ for some fixed $k \in \mathbb{N}$), as in Corollary 6.22. Since $m_L = O(L^2)$ and $d_L \sim L$ for the usual surface-code family [1, 4, 5], we have $m_L T_L = O(L^2 T_L)$, so for any fixed $\epsilon_{\mathrm{proc}} > 0$ the threshold

$$\delta_*(L,T_L;\epsilon_{\mathrm{proc}}) = \frac{2\epsilon_{\mathrm{proc}}}{m_L T_L} \tag{402}$$

is strictly positive for each finite L and decays at most polynomially in L. Thus, for any given $\delta > 0$, we can ensure $\delta < \delta_*(L,T_L;\epsilon_{\mathrm{proc}})$ by choosing δ sufficiently small (or, equivalently, by demanding sufficiently close Pauli error distributions for schedules A and B), and hence obtain

$$v_{\mathrm{Holo}}^{(L,T_L)} \;\ge\; 1 - \epsilon_{\mathrm{proc}}. \tag{403}$$

This is precisely the process part (ii) of Theorem 7.2, with $\delta_*(L,T) := \delta_*(L,T;\epsilon_{\mathrm{proc}})$.

Step 5: Combining logical and process parts.

We now combine Steps 3 and 4. Recall the decomposition

$$\epsilon_{\log} + \epsilon_{\mathrm{corr}} + \epsilon_{\mathrm{leak}} \;\le\; \epsilon,$$

and choose, in addition, $\epsilon_{\text{proc}} > 0$ with

$$\epsilon_{\log} + \epsilon_{\text{corr}} + \epsilon_{\text{leak}} + \epsilon_{\text{proc}} \ \le \ \epsilon.$$

For concreteness, one may take $\epsilon_{\log} = \epsilon_{\text{corr}} = \epsilon_{\text{leak}} = \epsilon_{\text{proc}} = \epsilon/4$, but any such partition suffices.

From Step 3, there exists L_* such that for all $L \ge L_*$ and $p < p_*$, $\varepsilon_{\text{ps}} < \varepsilon_{\text{ps},*}$ we have

$$1 - \upsilon_{\text{Th}}(\rho_{\text{out}}; \varphi_L^{(L)}) \le \epsilon_{\log} + \epsilon_{\text{corr}} + \epsilon_{\text{leak}} \le \epsilon.$$

From Step 4, for the chosen polynomial time horizon T_L and any $\delta < \delta_*(L, T_L; \epsilon_{\text{proc}})$, we have

$$1 - \upsilon_{\text{Holo}}^{(L,T_L)} \le \epsilon_{\text{proc}} \le \epsilon.$$

Thus, for any $\epsilon > 0$ we can choose:

- a code size $L \ge L_*$ and corresponding distance d_L;
- a polynomial time horizon $T = T_L = \text{poly}(d_L)$;
- physical noise parameters satisfying

$$p < p_*, \qquad \varepsilon_{\text{ps}} < \varepsilon_{\text{ps},*}, \qquad \delta < \delta_*(L, T_L; \epsilon_{\text{proc}}),$$

such that

$$\upsilon_{\text{Th}}(\rho_{\text{out}}; \varphi_L^{(L)}) \ \ge \ 1 - \epsilon, \qquad \upsilon_{\text{Holo}}^{(L,T_L)} \ \ge \ 1 - \epsilon. \tag{404}$$

In the Tetralemma semantics of the IQR framework (cf. [8]), this means that, for both the logical correctness proposition $\varphi_L^{(L)}$ and the process-equivalence proposition $\varphi_{\text{proc}}^{(L)}$, the Thesis corner carries weight at least $1 - \epsilon$, and the combined weight of the AntiThesis, Synthesis, and Holothesis corners is at most ϵ. This is the IQR analogue of fault-tolerant quantum computation in the standard stabilizer/topological-code setting [18, 13, 17, 2, 3, 1, 4, 5, 8], and completes the proof of Theorem 7.2. □

8 Phase 6: Concrete Worked Examples and Sanity Checks

In this section we illustrate the abstract IQR threshold machinery with small, explicit examples. The goal is to see concrete numbers and simple formulas that mirror the general structure of Phases 1–5 and the underlying IQR framework of [8].

We present three examples:

(a) a three-qubit repetition code in PQS / Twofold language;

(b) a tiny (toy) surface-code-like patch with 9 data qubits and a small stabilizer group;

(c) resource-ancilla (GHZ/cluster) usage inside the code and how its robustness folds into $(p, \delta, \varepsilon_{\text{ps}})$.

8.1 Example 1: 3-Qubit Repetition Code in PQS Language

8.1.1 Twofold fibres and encoding

Definition 8.1 (Three-site PQS patch and emergent register)**.** Let $\{x_1, x_2, x_3\} \subset \mathcal{U}$ be three PQS loci. At each x_j, the modal fibre decomposes as

$$\mathcal{H}_{\text{mod}}^{(\mathfrak{B}_{x_j})} = \mathcal{H}_{(0,1),x_j} \oplus \mathcal{H}_{(p,s),x_j},$$

with $\mathcal{H}_{(0,1),x_j} = \text{span}\{|0\rangle_{x_j}, |1\rangle_{x_j}\}$ the Twofold sector (as constructed in Section 3). The corresponding emergent three-qubit register is

$$\mathcal{H}_{\text{rep}} := \mathcal{H}_{(0,1),x_1} \otimes \mathcal{H}_{(0,1),x_2} \otimes \mathcal{H}_{(0,1),x_3} \;\cong\; (\mathbb{C}^2)^{\otimes 3},$$

which is the restriction of the general emergent register construction (Definition 5.2) to the three-site patch $\{x_1, x_2, x_3\}$.

Definition 8.2 (3-qubit repetition code)**.** The three-qubit repetition code (correcting single bit-flip errors) is defined by the code subspace $\mathcal{C}_{\text{rep}} \subset \mathcal{H}_{\text{rep}}$ with logical basis

$$|0_L\rangle := |0\rangle_{x_1}|0\rangle_{x_2}|0\rangle_{x_3}, \tag{405}$$

$$|1_L\rangle := |1\rangle_{x_1}|1\rangle_{x_2}|1\rangle_{x_3}, \tag{406}$$

in direct analogy with the standard three-qubit repetition / stabilizer-code construction [13, 5]. The stabilizer group is generated by the two commuting stabilizers

$$S_{\text{rep}} := \langle Z_1 Z_2,\ Z_2 Z_3 \rangle, \tag{407}$$

where Z_j denotes the Pauli Z acting on the Twofold fibre $\mathcal{H}_{(0,1),x_j}$. The corresponding code projector is

$$P_{\text{rep}} := \frac{1}{4}\Big(\mathbb{I} + Z_1 Z_2\Big)\Big(\mathbb{I} + Z_2 Z_3\Big), \tag{408}$$

which projects onto the joint +1 eigenspace of $Z_1 Z_2$ and $Z_2 Z_3$.

8.1.2 Local duality and bit-flip noise from $\hat{G}_{(0,1)}$

Definition 8.3 (Local duality operator)**.** At each site x_j, define the *local duality* on the Twofold fibre by the Pauli X:

$$D_{x_j} := X_j := |0\rangle_{x_j}\langle 1|_{x_j} + |1\rangle_{x_j}\langle 0|_{x_j}. \tag{409}$$

This exchanges the Thesis and Antithesis basis states in the Twofold sector, consistent with the modal interpretation of the Fourfold basis in the IQR framework (Section 3 and [8]).

Assumption 8.4 (Bit-flip noise from the Twofold correlator)**.** *Assume that the Twofold block of the noise kernel $\Delta\hat{G}_{(0,1)}$ at each x_j generates, to leading order in the IQR noise parameters, an effective* bit-flip *Pauli channel on $\mathcal{H}_{(0,1),x_j}$:*

$$\Lambda_p^{(X)}(\rho) := (1-p)\,\rho + p\,X\rho X, \qquad 0 \le p \ll 1, \tag{410}$$

in analogy with standard bit-flip noise models for toy codes and repetition codes in quantum error correction [18, 20, 5]. In the Pauli-channel parameterization of Definition 5.11, this corresponds to Pauli probabilities

$$q(\mathbb{I}) = 1 - p, \quad q(X) = p, \quad q(Y) = 0, \quad q(Z) = 0.$$

We further assume independent and identically distributed noise across the three PQS, so that the register-level noise on $\mathcal{H}_{\text{rep}}$ is

$$\Lambda_p^{(3)} := \left(\Lambda_p^{(X)}\right)^{\otimes 3}, \tag{411}$$

consistent with the independent local-Pauli noise assumptions used in the general threshold analysis [1, 4, 5].

8.1.3 One-cycle logical error and Thesis valuation

We now model a single QEC cycle for the three-qubit repetition code under bit-flip noise induced from the Twofold correlator, and compute the resulting logical error probability and Thesis valuation in this toy setting, in direct analogy with the general framework of Phases 2–3 and standard textbook treatments [18, 20, 13, 5].

Definition 8.5 (QEC cycle for the 3-qubit repetition code)**.** A single QEC cycle for the three-qubit repetition code $\mathcal{C}_{\text{rep}}$ consists of:

(a) independent bit-flip noise $\Lambda_p^{(3)}$ on the emergent register $\mathcal{H}_{\text{rep}}$ as in Assumption 8.4;

(b) measurement of the two stabilizers $Z_1 Z_2$ and $Z_2 Z_3$;

(c) classical decoding of the syndrome and application of a single-qubit X-correction on the majority-vote rule: the unique qubit identified as flipped (if any) is corrected by X.

We assume ideal (noise-free) stabilizer measurement and classical decoding, so that the only physical noise per cycle is the bit-flip channel $\Lambda_p^{(3)}$ on the data qubits, as in standard toy models of QEC cycles [18, 20, 5].

Definition 8.6 (Logical error probability for the repetition code)**.** Let $\mathcal{N}_p^{(\text{rep})}$ denote the overall CPTP map on the logical subspace $\mathcal{C}_{\text{rep}}$ obtained by the QEC cycle of Definition 8.5 composed with the embedding of $\mathcal{C}_{\text{rep}}$ into $\mathcal{H}_{\text{rep}}$. We define the *logical error probability* $p_L^{(\text{rep})}$ as

$$p_L^{(\text{rep})} := \sup_{\rho_{\log} \in \mathcal{S}(\mathcal{C}_{\text{rep}})} \left(1 - \operatorname{Tr}\left[P_{\text{rep}} \mathcal{N}_p^{(\text{rep})}(\rho_{\log})\right]\right), \tag{412}$$

i.e. the worst-case probability that the output state lies outside the ideal logical subspace or has undergone a logical bit flip. In this simple code, this coincides with the probability that the net effect of physical bit flips and corrections differs from the identity on the logical basis $\{|0_L\rangle, |1_L\rangle\}$ [13, 5].

Proposition 8.7 (Logical error probability for 3-qubit repetition)**.** *Under Assumption 8.4 and the QEC cycle of Definition 8.5, the logical error probability per cycle for the three-qubit repetition code is*

$$p_L^{(\mathrm{rep})} = 3p^2 - 2p^3 = \mathbb{P}(\textit{at least two bit flips on the three PQS}). \tag{413}$$

In particular, for $0 \le p \ll 1$ *one has*

$$p_L^{(\mathrm{rep})} = 3p^2 + O(p^3), \tag{414}$$

*so the logical error rate scales quadratically in the physical bit-flip rate, as expected for a distance-*3 *code [18, 20, 13, 5].*

Proof. Because the noise is i.i.d. across the three PQS, the bit-flip pattern on $\{x_1, x_2, x_3\}$ is given by three independent Bernoulli variables with parameter p. There are four types of error patterns, with probabilities:

$$\text{(i) No error (0 flips):} \quad \mathbb{P}_0 = (1-p)^3, \tag{415}$$

$$\text{(ii) Exactly one error:} \quad \mathbb{P}_1 = 3p(1-p)^2, \tag{416}$$

$$\text{(iii) Exactly two errors:} \quad \mathbb{P}_2 = 3p^2(1-p), \tag{417}$$

$$\text{(iv) Three errors:} \quad \mathbb{P}_3 = p^3. \tag{418}$$

The repetition-code decoder implements majority vote: if at most one qubit is flipped, the syndrome uniquely identifies the flipped qubit (or no flip), and a suitable X-correction recovers the correct logical state; if two or three qubits are flipped, the majority is wrong and the decoder applies either no correction or a single-qubit correction that produces a net logical bit flip.

More concretely:

- For 0 flips, the code state is unchanged and the decoder does nothing: there is no logical error.
- For exactly 1 flip, the syndrome identifies the flipped site, and the decoder applies X on that site, undoing the error: there is no logical error.
- For exactly 2 flips, the two flipped qubits form the majority, so the decoder erroneously flips the third (unflipped) qubit: the net effect on the logical state is a logical X_L (bit flip), i.e. a logical error.
- For 3 flips, X is applied to all qubits; in the computational basis, this also implements a logical X_L on $|0_L\rangle \leftrightarrow |1_L\rangle$, so this is likewise a logical error.

Hence the logical error probability is

$$p_L^{(\mathrm{rep})} = \mathbb{P}_2 + \mathbb{P}_3 = 3p^2(1-p) + p^3 = 3p^2 - 2p^3. \tag{419}$$

For $0 \le p \ll 1$ we expand

$$3p^2 - 2p^3 = 3p^2 + O(p^3), \tag{420}$$

showing the expected $O(p^2)$ scaling for a distance-3 repetition code. □

Definition 8.8 (Thesis valuation for the repetition-code example)**.** In analogy with the general logical Thesis valuation (Definition 2.9), we define the logical correctness proposition for the repetition code, φ_{rep}, to assert that "no logical bit flip has occurred during the QEC cycle," and set

$$\upsilon_{\text{Th}}\Big(\rho_{\text{out}}; \varphi_{\text{rep}}\Big) := 1 - p_L^{(\text{rep})}. \tag{421}$$

Corollary 8.9 (Thesis valuation for the 3-qubit repetition code)**.** *Under Assumption 8.4 and the QEC cycle of Definition 8.5, the Thesis valuation of the logical correctness proposition* φ_{rep} *satisfies*

$$\upsilon_{\text{Th}}\Big(\rho_{\text{out}}; \varphi_{\text{rep}}\Big) = 1 - (3p^2 - 2p^3) = 1 - 3p^2 + 2p^3. \tag{422}$$

In particular, for $0 \le p \ll 1$*,*

$$\upsilon_{\text{Th}}\Big(\rho_{\text{out}}; \varphi_{\text{rep}}\Big) = 1 - 3p^2 + O(p^3), \tag{423}$$

so that the deficit of the Thesis valuation scales as $1 - \upsilon_{\text{Th}}(\rho_{\text{out}}; \varphi_{\text{rep}}) = O(p^2)$*, in direct analogy with the distance-controlled scaling in the full surface-code threshold theorem of Phase 3, specialized here to a code of distance* $d = 3$*.*

Proof. By Definition 8.8, the logical correctness proposition φ_{rep} for the three-qubit repetition code is the statement that "no logical bit flip has occurred during the QEC cycle," and its Thesis valuation is defined as

$$\upsilon_{\text{Th}}\Big(\rho_{\text{out}}; \varphi_{\text{rep}}\Big) := 1 - p_L^{(\text{rep})}, \tag{424}$$

where $p_L^{(\text{rep})}$ is the logical error probability per cycle.

Under Assumption 8.4 and the QEC cycle of Definition 8.5, Proposition 8.7 gives

$$p_L^{(\text{rep})} = 3p^2 - 2p^3. \tag{425}$$

Substituting this into the definition of the Thesis valuation yields

$$\upsilon_{\text{Th}}\Big(\rho_{\text{out}}; \varphi_{\text{rep}}\Big) = 1 - p_L^{(\text{rep})} = 1 - (3p^2 - 2p^3) = 1 - 3p^2 + 2p^3, \tag{426}$$

which is the claimed exact expression.

For the small-noise regime $0 \le p \ll 1$, we note that $2p^3 = O(p^3)$, so

$$\upsilon_{\text{Th}}\Big(\rho_{\text{out}}; \varphi_{\text{rep}}\Big) = 1 - 3p^2 + O(p^3). \tag{427}$$

Equivalently, the deficit of the Thesis valuation is

$$1 - \upsilon_{\text{Th}}\Big(\rho_{\text{out}}; \varphi_{\text{rep}}\Big) = p_L^{(\text{rep})} = 3p^2 - 2p^3 = p^2(3 - 2p). \tag{428}$$

For $0 \le p \le 1$ we have $0 \le 3 - 2p \le 3$, hence

$$0 \ \le \ 1 - \upsilon_{\text{Th}}\Big(\rho_{\text{out}}; \varphi_{\text{rep}}\Big) = p^2(3 - 2p) \ \le \ 3p^2. \tag{429}$$

Thus there exists a constant $C = 3$ such that

$$1 - \upsilon_{\text{Th}}\Big(\rho_{\text{out}}; \varphi_{\text{rep}}\Big) \le C\, p^2 \tag{430}$$

for all sufficiently small p, which is precisely the statement

$$1 - \upsilon_{\mathrm{Th}}\Big(\rho_{\mathrm{out}}; \varphi_{\mathrm{rep}}\Big) = O(p^2).$$

This matches the expected distance-controlled scaling for a distance-3 code in the small-noise regime [18, 20, 13, 5] and is the repetition-code analogue of the exponential-in-d_L suppression in the full surface-code threshold theorem of Phase 3. □

Remark 8.10. The three-qubit repetition code thus provides a minimal concrete instance of the general IQR threshold logic: the emergent Twofold fibres realize the physical qubits, the bit-flip noise channel is derived from the Twofold block of the correlator $\Delta\hat{G}_{(0,1)}$, and the logical Thesis valuation matches the standard QEC intuition that a distance-d code suppresses logical errors from $O(p)$ to $O(p^{\lceil d/2 \rceil})$ in the small-noise regime [18, 20, 13, 5]. In the full surface-code family, this quadratic suppression is replaced by an exponential-in-d_L suppression as captured by Theorem 5.28 and Corollary 5.29.

8.1.4 Holosymmetry for two correction gadgets

Now consider two different correction gadgets A and B implementing the same logical repetition-code cycle (e.g. differing in stabilizer measurement order, ancilla layout, or classical decoding circuitry), in the sense of the dual-history comparison framework introduced in Section 6.

Assumption 8.11 (Pauli error distributions for two repetition gadgets)**.** *For a single cycle, suppose:*

- *Schedule A induces single-site bit-flip distributions* $\mathbf{q}^{(A)} = (q^{(A)}(\mathbb{I}), q^{(A)}(X), 0, 0) = (1 - p_A, p_A, 0, 0)$*;*
- *Schedule B induces* $\mathbf{q}^{(B)} = (1 - p_B, p_B, 0, 0)$*.*

We assume the same p_A, p_B on all three sites. The error rates p_A, p_B are determined by the corresponding deviations of $\hat{G}_{(0,1)}$ along each schedule (as in the IQR-to-Pauli reduction of Section 4).

Definition 8.12 (Single-site Pauli overlap for repetition gadgets)**.** The single-site overlap is

$$k := \sum_{P \in \{\mathbb{I}, X, Y, Z\}} \sqrt{q^{(A)}(P)\, q^{(B)}(P)} = \sqrt{(1 - p_A)(1 - p_B)} + \sqrt{p_A p_B}. \tag{431}$$

Define the differential parameter $\delta := 1 - k$ as in Definition 6.14.

Lemma 8.13 (Holosymmetric valuation for the 3-qubit cycle)**.** *Let $m_3 = 3$ be the number of PQS touched by the gadgets (all three qubits). Then, by the Pauli-channel holosymmetry theorem (Theorem 6.12),*

$$\upsilon^{(\mathrm{rep})}_{\mathrm{Holo}} = \frac{1}{2}\Big(1 + k^{m_3}\Big) = \frac{1}{2}\Big(1 + k^3\Big). \tag{432}$$

In particular, for $\delta = 1 - k \in [0, 1]$,

$$1 - \upsilon^{(\mathrm{rep})}_{\mathrm{Holo}} = \frac{1}{2}\Big(1 - (1 - \delta)^3\Big) \;\le\; \frac{3}{2}\,\delta, \tag{433}$$

in agreement with the general single-cycle bound of Corollary 6.15.

Proof. By Assumption 8.11, the noise induced by schedule A on each of the three PQS $\{x_1, x_2, x_3\}$ is a single-qubit Pauli channel with distribution

$$\mathbf{q}^{(A)} = (1 - p_A, p_A, 0, 0),$$

and similarly for schedule B with distribution $\mathbf{q}^{(B)} = (1 - p_B, p_B, 0, 0)$. Since these distributions are the same on all three sites and the noise is independent across sites, we are in the homogeneous, independent Pauli regime of Assumption 6.10 with

$$Q_L = \{x_1, x_2, x_3\}, \qquad m_3 := |Q_L| = 3,$$

and $k_x \equiv k$ for all $x \in Q_L$, where k is given by Definition 8.12. In particular, k is the (classical) Bhattacharyya overlap of the two single-site Pauli error distributions, so $0 \le k \le 1$ and hence $\delta = 1 - k \in [0, 1]$.

The Pauli-channel holosymmetry theorem (Theorem 6.12) states that, for such independent single-qubit Pauli noise on the set Q_L of touched PQS, the holosymmetric process overlap factorizes as

$$C^{(L)} = \prod_{x \in Q_L} k_x. \tag{434}$$

In the homogeneous case $k_x \equiv k$, this becomes

$$C^{(L)} = k^{m_L}. \tag{435}$$

Specializing to the present repetition-code setting, where $m_L = m_3 = 3$, we obtain

$$C^{(\mathrm{rep})} = \prod_{j=1}^{3} k = k^3. \tag{436}$$

By Definition 6.4, the single-cycle holosymmetric valuation is related to the overlap by

$$v_{\mathrm{Holo}}^{(\mathrm{rep})} = \frac{1}{2}\Big(1 + C^{(\mathrm{rep})}\Big) = \frac{1}{2}\Big(1 + k^3\Big), \tag{437}$$

which establishes the first claim.

For the bound in terms of the differential parameter $\delta = 1 - k$, we write

$$1 - v_{\mathrm{Holo}}^{(\mathrm{rep})} = \frac{1}{2}\Big(1 - k^3\Big) = \frac{1}{2}\Big(1 - (1 - \delta)^3\Big). \tag{438}$$

For $\delta \in [0, 1]$, the Bernoulli inequality (or the binomial expansion with positivity of higher-order terms) gives

$$(1 - \delta)^3 = 1 - 3\delta + 3\delta^2 - \delta^3 \;\ge\; 1 - 3\delta, \tag{439}$$

since $3\delta^2 - \delta^3 \ge 0$ on $[0, 1]$. Rearranging,

$$1 - (1 - \delta)^3 \;\le\; 3\delta. \tag{440}$$

Substituting into the expression for $1 - v_{\mathrm{Holo}}^{(\mathrm{rep})}$ yields

$$1 - v_{\mathrm{Holo}}^{(\mathrm{rep})} = \frac{1}{2}\Big(1 - (1 - \delta)^3\Big) \;\le\; \frac{1}{2} \cdot 3\delta = \frac{3}{2}\,\delta. \tag{441}$$

This is exactly the specialization of the general single-cycle bound of Corollary 6.15,

$$1 - v_{\mathrm{Holo}}^{(L)} \le \frac{1}{2} m_L \delta,$$

to the case $m_L = 3$. Hence the lemma is proved and the 3-qubit example is fully consistent with the general Pauli-channel holosymmetry framework of Phase 4. ☐

Example 8.14 (Numerical sanity check)**.** Let $p_A = 0.01$ and $p_B = 0.012$. Then

$$k = \sqrt{0.99 \cdot 0.988} + \sqrt{0.01 \cdot 0.012} \approx 0.989 + 0.011 \approx 1.000,$$

so $\delta = 1 - k \approx 0$ (to three decimal places), and

$$v_{\mathrm{Holo}}^{(\mathrm{rep})} = \frac{1}{2}\Big(1 + k^3\Big) \approx \frac{1}{2}(1 + 1^3) = 1.$$

Thus, for such close error rates, the two gadgets are essentially perfectly holosymmetric at the repetition-code scale, providing a concrete sanity check of the general IQR holosymmetry formulas in a minimal setting.

8.2 Example 2: Tiny 9-Qubit Surface-Code-Like Patch

We now consider a small, explicit stabilizer system on a 3 × 3 emergent lattice. For simplicity, we focus on the Z-stabilizer sector under pure Z-type noise; the X-sector behaves analogously. This serves as a concrete sanity check for the stabilizer robustness polynomial $F_{\mathrm{SC},L}$ (Definition 5.14) and its small-λ behaviour, in direct analogy with standard stabilizer and surface-code constructions [13, 1, 15, 5].

8.2.1 Patch and Z-stabilizers

Definition 8.15 (9-qubit patch and indices)**.** Let $\{x_{i,j} \mid i, j \in \{1, 2, 3\}\} \subset \mathcal{U}$ be a 3×3 grid of PQS loci, with emergent Twofold qubits at each site (as in Section 3 and [8]). Label the corresponding data qubits by $1, \dots, 9$ in row-major order:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

We restrict attention to Z-type stabilizers acting on these 9 qubits, in direct analogy with the Z-plaquettes of a planar surface code [1, 14, 15, 5].

Definition 8.16 (Plaquette Z-stabilizers)**.** Define four Z-type plaquette stabilizers:

$$S_1 := Z_1 Z_2 Z_4 Z_5, \tag{442}$$

$$S_2 := Z_2 Z_3 Z_5 Z_6, \tag{443}$$

$$S_3 := Z_4 Z_5 Z_7 Z_8, \tag{444}$$

$$S_4 := Z_5 Z_6 Z_8 Z_9. \tag{445}$$

Each S_i is a tensor product of single-qubit Pauli Z operators (and identities) on the 9-qubit register. Since all local factors are Pauli Z, all S_i commute pairwise, i.e. $[S_i, S_j] = 0$ for all i, j, in accordance with the general stabilizer-code construction where commuting checks are obtained from products of compatible local Paulis on a lattice [13, 1, 5].

Let

$$S_Z := \langle S_1, S_2, S_3, S_4 \rangle \tag{446}$$

be the abelian stabilizer subgroup they generate. The four generators are independent: if $S_1^{a_1} S_2^{a_2} S_3^{a_3} S_4^{a_4} = \mathbb{I}$ with $a_i \in \{0, 1\}$, then looking at the support on qubits $1, 3, 7, 9$ shows $a_1 = a_2 = a_3 = a_4 = 0$ (since Z_1 appears only in S_1, Z_3 only in S_2, Z_7 only in S_3, and Z_9 only in S_4). Hence $S_Z \cong (\mathbb{Z}_2)^4$ and

$$|S_Z| = 2^4 = 16, \tag{447}$$

in agreement with the standard stabilizer-group counting $|S| = 2^{n-k}$ for an abelian subgroup of the n-qubit Pauli group with k logical qubits [13, 5].

6.2.2. Explicit weight distribution in S_Z

For any $g \in S_Z$, write $g = \bigotimes_{j=1}^{9} P_j$ with $P_j \in \{\mathbb{I}, Z, -Z\}$. The Z-weight of g is

$$w_Z(g) := \#\{j : P_j \in \{Z, -Z\}\}. \tag{448}$$

Because all S_i are Z-type, any product g is again Z-type (up to an overall phase), and its support is the symmetric difference of the supports of the S_i appearing in the product, as is standard for stabilizer products in the Pauli group [13, 5].

Let $s_i \subset \{1, \ldots, 9\}$ be the support of S_i:

$$s_1 = \{1, 2, 4, 5\}, \tag{449}$$

$$s_2 = \{2, 3, 5, 6\}, \tag{450}$$

$$s_3 = \{4, 5, 7, 8\}, \tag{451}$$

$$s_4 = \{5, 6, 8, 9\}. \tag{452}$$

Products of stabilizers correspond to symmetric differences of these sets: if $g = S_1^{a_1} S_2^{a_2} S_3^{a_3} S_4^{a_4}$ with $a_i \in \{0, 1\}$, then the support of g is

$$\operatorname{supp}(g) = s_1^{(a_1)} \Delta s_2^{(a_2)} \Delta s_3^{(a_3)} \Delta s_4^{(a_4)},$$

where $s_i^{(0)} := \varnothing$ and $s_i^{(1)} := s_i$, and Δ denotes symmetric difference.

Lemma 8.17 (Weight distribution in S_Z). *The stabilizer group S_Z has the following Z-weight distribution:*

- *one element of weight* 0 *(the identity);*
- *nine elements of weight* 4*;*
- *six elements of weight* 6.

In total, $1 + 9 + 6 = 16$ *elements, as required.*

Proof. There are $2^4 = 16$ elements in S_Z, corresponding to all choices of $a = (a_1, a_2, a_3, a_4) \in \{0,1\}^4$ in $g(a) = S_1^{a_1} S_2^{a_2} S_3^{a_3} S_4^{a_4}$. We classify these by the Hamming weight of a and compute the symmetric differences of supports.

(1) Identity. For $a = (0,0,0,0)$ we have $g = \mathbb{I}$ with empty support and

$$w_Z(\mathbb{I}) = 0.$$

This gives the unique weight-0 element.

(2) Single generators. For a with Hamming weight 1 we obtain the four generators:

$$S_1,\ S_2,\ S_3,\ S_4,$$

each with support s_i of size 4. Hence

$$w_Z(S_i) = |s_i| = 4, \qquad i = 1,2,3,4.$$

(3) Pairwise products. For a with Hamming weight 2, say $a_i = a_j = 1$ and others 0, the support is $s_i \Delta s_j$. A direct computation yields:

$$s_1\Delta s_2 = \{1,2,4,5\}\Delta\{2,3,5,6\} = \{1,3,4,6\}, \qquad |s_1\Delta s_2| = 4, \tag{453}$$

$$s_1\Delta s_3 = \{1,2,4,5\}\Delta\{4,5,7,8\} = \{1,2,7,8\}, \qquad |s_1\Delta s_3| = 4, \tag{454}$$

$$s_1\Delta s_4 = \{1,2,4,5\}\Delta\{5,6,8,9\} = \{1,2,4,6,8,9\}, \qquad |s_1\Delta s_4| = 6, \tag{455}$$

$$s_2\Delta s_3 = \{2,3,5,6\}\Delta\{4,5,7,8\} = \{2,3,4,6,7,8\}, \qquad |s_2\Delta s_3| = 6, \tag{456}$$

$$s_2\Delta s_4 = \{2,3,5,6\}\Delta\{5,6,8,9\} = \{2,3,8,9\}, \qquad |s_2\Delta s_4| = 4, \tag{457}$$

$$s_3\Delta s_4 = \{4,5,7,8\}\Delta\{5,6,8,9\} = \{4,6,7,9\}, \qquad |s_3\Delta s_4| = 4. \tag{458}$$

Thus among the six pairwise products there are four elements of weight 4 and two of weight 6.

(4) Triple products. For a of Hamming weight 3, say $a_i = a_j = a_k = 1$, the support is $s_i\Delta s_j\Delta s_k$. We may compute these as symmetric differences of the pairwise supports above with the remaining s_ℓ.

For example,

$$s_1\Delta s_2\Delta s_3 = (s_1\Delta s_2)\Delta s_3 = \{1,3,4,6\}\Delta\{4,5,7,8\} = \{1,3,5,6,7,8\}, \tag{459}$$

which has weight 6. Similarly,

$$s_1\Delta s_2\Delta s_4 = \{1,3,4,6\}\Delta\{5,6,8,9\} = \{1,3,4,5,8,9\}, \tag{460}$$

$$s_1\Delta s_3\Delta s_4 = \{1,2,7,8\}\Delta\{5,6,8,9\} = \{1,2,5,6,7,9\}, \tag{461}$$

$$s_2\Delta s_3\Delta s_4 = \{2,3,4,6,7,8\}\Delta\{5,6,8,9\} = \{2,3,4,5,7,9\}, \tag{462}$$

all of which also have weight 6. Hence all four triple products have $w_Z = 6$.

(5) Quadruple product. Finally, for $a = (1,1,1,1)$ the support is

$$s_1\Delta s_2\Delta s_3\Delta s_4 = (s_1\Delta s_2\Delta s_3)\Delta s_4 \tag{463}$$

$$= \{1,3,5,6,7,8\}\Delta\{5,6,8,9\} = \{1,3,7,9\}, \tag{464}$$

which has weight 4.

Collecting all cases:

- one element of weight 0 (identity);
- four generators of weight 4;
- four pairwise products of weight 4;
- one quadruple product of weight 4;
- two pairwise products of weight 6;
- four triple products of weight 6.

Hence

$$\#\{w_Z = 0\} = 1, \qquad \#\{w_Z = 4\} = 4 + 4 + 1 = 9, \qquad \#\{w_Z = 6\} = 2 + 4 = 6,$$

and no other weights occur. This exhausts all $2^4 = 16$ elements of S_Z and yields the stated distribution, providing a concrete instance of the general stabilizer weight-counting framework used in Proposition 5.18 and related stabilizer-code analyses [19, 13, 5]. □

8.2.2 Robustness polynomial under Z-type Pauli noise

Consider a single-qubit Z-only Pauli channel

$$\Lambda^{(Z)}_{p_Z}(\rho) := (1 - p_Z)\,\rho + p_Z\, Z\rho Z, \tag{465}$$

with shrink factor for Z:

$$\lambda_Z = 1 - 2p_Z. \tag{466}$$

On Z-type Pauli operators, the channel acts by $\Lambda^{(Z)}_{p_Z}(Z) = \lambda_Z Z$. On a many-body Z-type stabilizer g with Z-weight $w_Z(g)$, we therefore have

$$\Lambda^{\otimes 9}_{p_Z}(g) = \lambda_Z^{w_Z(g)}\, g, \tag{467}$$

exactly as in the Pauli-covariant setting of Definition 5.11 and Proposition 5.18 for general stabilizer codes [19, 13, 5].

Definition 8.18 (9-qubit Z-sector robustness polynomial)**.** Restricting to the Z-sector stabilizer group S_Z of Definition 8.16, define

$$F_{\mathrm{SC},Z}(\lambda_Z) := \frac{1}{|S_Z|} \sum_{g \in S_Z} \lambda_Z^{w_Z(g)}. \tag{468}$$

This is the Z-sector analogue of the full surface-code robustness functional $F_{\mathrm{SC},L}$ of Definition 5.14, specialized to pure Z-type Pauli noise.

Proposition 8.19 (Explicit $F_{\mathrm{SC},Z}(\lambda_Z)$)**.** *Using Lemma 8.17, the robustness polynomial in the Z-sector is*

$$F_{\mathrm{SC},Z}(\lambda_Z) = \frac{1}{16}\Big(1 + 9\,\lambda_Z^4 + 6\,\lambda_Z^6\Big). \tag{469}$$

Proof. By Lemma 8.17, the stabilizer group S_Z has: one element of weight 0 (the identity), nine elements of weight 4, and six elements of weight 6, and no other weights occur. Substituting this distribution into the definition of $F_{\text{SC},Z}(\lambda_Z)$ gives

$$F_{\text{SC},Z}(\lambda_Z) = \frac{1}{|S_Z|} \sum_{g \in S_Z} \lambda_Z^{w_Z(g)} = \frac{1}{16}\Big(1 \cdot \lambda_Z^0 + 9 \cdot \lambda_Z^4 + 6 \cdot \lambda_Z^6\Big), \tag{470}$$

as claimed. This is a concrete instance of the general polynomial structure of stabilizer robustness functionals from Proposition 5.18 [19, 13, 5]. □

Example 8.20 (Small-p_Z expansion and numerical check)**.** For small p_Z we write $\lambda_Z = 1 - 2p_Z$ and expand to first order:

$$(1 - 2p_Z)^4 = 1 - 8p_Z + O(p_Z^2), \tag{471}$$

$$(1 - 2p_Z)^6 = 1 - 12p_Z + O(p_Z^2). \tag{472}$$

Substituting into $F_{\text{SC},Z}$,

$$F_{\text{SC},Z}(\lambda_Z) = \frac{1}{16}\Big(1 + 9(1 - 8p_Z) + 6(1 - 12p_Z)\Big) + O(p_Z^2) \tag{473}$$

$$= \frac{1}{16}\Big(1 + 9 + 6 - (72p_Z + 72p_Z)\Big) + O(p_Z^2) \tag{474}$$

$$= 1 - 9p_Z + O(p_Z^2). \tag{475}$$

Thus the first-order decay rate of the robustness functional is set by the average Z-weight of stabilizers in S_Z, in accordance with the general moment interpretation of robustness polynomials [13, 5].

For a concrete numerical value, take $p_Z = 0.01$, so $\lambda_Z = 0.98$. Then

$$F_{\text{SC},Z}(0.98) = \frac{1}{16}\Big(1 + 9 \cdot 0.98^4 + 6 \cdot 0.98^6\Big) \approx 0.91, \tag{476}$$

showing a modest reduction in stabilizer robustness at the 1% single-qubit Z-error level for this tiny patch, consistent with general stabilizer-code behaviour under Pauli noise [13, 1, 15, 5].

8.2.3 Logical error and holosymmetry on the tiny patch

A standard distance-3 rotated planar surface code with 9 data qubits (plus syndrome ancillas) encodes one logical qubit and has code distance $d_L = 3$, in line with minimal rotated planar-code constructions [1, 14, 4, 5]. Under independent single-qubit Pauli noise of physical rate p per data qubit, the minimal weight of any nontrivial logical operator is d_L, and the code corrects all error patterns of weight at most

$$t := \left\lfloor \frac{d_L - 1}{2} \right\rfloor = 1,$$

so uncorrectable patterns have weight at least $t + 1 = (d_L + 1)/2 = 2$, in agreement with the standard distance-based intuition for stabilizer codes [13, 5].

Lemma 8.21 (Logical error scaling for the tiny distance-3 patch)**.** *Let p be the physical Pauli error rate per data qubit on the 9-qubit distance-3 rotated planar patch, with independent noise across qubits and an efficient decoder that corrects all weight-≤ 1 error patterns. Then the logical error probability per cycle satisfies*

$$p_L(p) = \Theta\Big(p^{(d_L+1)/2}\Big) = \Theta(p^2), \tag{477}$$

for $0 \leq p \ll 1$. More explicitly, there is a constant $C' > 0$, depending on the geometry and the decoder, such that

$$p_L(p) \sim C'\, p^{(d_L+1)/2} = C'\, p^2, \tag{478}$$

as $p \to 0$, reflecting the standard small-distance surface-code scaling under local noise [1, 4, 5].

Proof. Because the code has distance $d_L = 3$, any nontrivial logical Pauli operator must have weight at least 3; conversely, there exist weight-3 logical operators (logical strings across the patch), as in the usual rotated planar surface code [1, 14, 4, 5]. A decoder that corrects all error patterns of weight ≤ 1 fails only when the physical error pattern either *(i)* has weight at least 2, or *(ii)* produces a syndrome that is degenerate between a correctable pattern and a logical pattern. In either case, the minimal Hamming weight of error patterns that lead to a logical failure is at least $2 = (d_L + 1)/2$.

Under independent Pauli noise of rate p per qubit, the probability of any specific weight-w pattern scales as $p^w(1-p)^{9-w}$; hence the total probability of all weight-w patterns is $\Theta(p^w)$ for fixed w as $p \to 0$. Since all weight-1 patterns are correctable by assumption, and there exist weight-2 patterns that lead to logical failure (e.g. pairs forming halves of a logical string, depending on the decoder), we have

$$p_L(p) = \Theta(p^2), \tag{479}$$

with the proportionality constant C' counting, in effect, how many weight-2 (and higher) patterns are decoded as logical errors rather than successfully corrected, in line with the usual distance-based error counting in stabilizer codes [13, 5]. This establishes the claimed asymptotic form. □

Example 8.22 (Toy logical error estimate)**.** Assuming a modest $p_Z = 1\%$ and a decoder with an effective constant $C' \approx 10$ (counting the relevant low-weight uncorrectable patterns), we obtain the toy estimate

$$p_L(p_Z) \sim 10\,(0.01)^2 = 10^{-3},$$

so the logical Thesis valuation $v_{\text{Th}}(\rho_{\text{out}}; \varphi_L^{(L)}) = 1 - p_L(p_Z)$ is roughly

$$v_{\text{Th}} \approx 1 - 10^{-3} = 0.999,$$

in qualitative agreement with the logical scaling estimates of Theorem 5.28 specialized to a small-distance surface-code instance [1, 4, 5].

For process holosymmetry, suppose two syndrome-extraction schedules A and B for this tiny patch differ slightly in how they couple to the environment, leading to slightly different Z-error rates $p_Z^{(A)}$ and $p_Z^{(B)}$, and hence slightly different Z-Pauli distributions. Their single-site overlap k and differential parameter $\delta = 1 - k$ then enter the Pauli-channel holosymmetry formula as in Theorem 6.12 and Corollary 6.15.

Let m_L denote the number of PQS touched per cycle (data qubits plus ancillas); for this tiny patch one can take $m_L = O(9)$, consistent with the scale that appears in the process-threshold condition of Theorem 6.21.

Example 8.23 (Process holosymmetry on the 9-qubit patch)**.** Suppose

$$p_Z^{(A)} = 0.01, \qquad p_Z^{(B)} = 0.012.$$

Ignoring X/Y components for simplicity (pure Z-sector noise), the single-site Pauli overlap is

$$k = \sum_{P \in \{\mathbb{I}, Z\}} \sqrt{q^{(A)}(P)\, q^{(B)}(P)} = \sqrt{(1 - p_Z^{(A)})(1 - p_Z^{(B)})} + \sqrt{p_Z^{(A)} p_Z^{(B)}}. \tag{480}$$

Numerically,

$$k \approx \sqrt{0.99 \cdot 0.988} + \sqrt{0.01 \cdot 0.012} \approx 0.989 + 0.011 \approx 1.000,$$

so $\delta = 1 - k \approx 0$ on the scale of a few 10^{-5}. If $m_L \approx 9$, Theorem 6.12 gives the single-cycle holosymmetric valuation

$$v_{\text{Holo}}^{(L)} = \frac{1}{2}\Big(1 + k^{m_L}\Big) \approx \frac{1}{2}(1 + 1^9) = 1, \tag{481}$$

and, under stationarity and independence across cycles (Assumption 6.17 and Lemma 6.19), the T-cycle valuation is

$$v_{\text{Holo}}^{(L,T)} = \frac{1}{2}\Big(1 + k^{m_L T}\Big) \approx 1, \tag{482}$$

as long as $m_L T\, \delta \ll 1$. This is precisely the condition appearing in the process-threshold bound of Theorem 6.21, and thus the tiny 9-qubit patch provides a concrete sanity check of the general Phase 4 holosymmetry analysis in a minimal surface-code-like setting.

8.3 Example 3: Resource Ancillas (GHZ/Cluster) Inside the Code

We finally consider resource states (GHZ or small cluster states) used inside the QEC circuit, e.g. as ancillas for syndrome extraction or lattice surgery, and show how their robustness and holosymmetry feed into the effective parameters $(p, \delta, \varepsilon_{\text{ps}})$ of the threshold theorem, in line with standard stabilizer-based ancilla usage in fault-tolerant architectures [13, 1, 12, 5].

8.3.1 GHZ ancillas and robustness

Definition 8.24 (n-qubit GHZ state)**.** The n-qubit GHZ state on an emergent register $\mathcal{H}_{\text{GHZ}} := (\mathbb{C}^2)^{\otimes n}$ is

$$|\text{GHZ}_n\rangle := \frac{1}{\sqrt{2}}\Big(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\Big), \tag{483}$$

the standard Greenberger–Horne–Zeilinger state [26, 27]. Its stabilizer group is generated by

- the global $X^{\otimes n}$ operator;
- the pairwise $Z_i Z_{i+1}$ for $i = 1, \dots, n-1$,

in direct analogy with the stabilizer formalism of [13].

Assumption 8.25 (Local Pauli noise on GHZ ancilla)**.** *Suppose a GHZ ancilla of size n is prepared in the IQR arena and then subject to i.i.d. single-qubit Pauli noise with channel Λ having probabilities $\{q_P\}_{P\in\{\mathbb{I},X,Y,Z\}}$ and shrink factors $\lambda_X, \lambda_Y, \lambda_Z$, as in Definition 5.11 and [6, 5].*

Definition 8.26 (GHZ robustness functional)**.** Let $P_{\mathrm{GHZ}} := |\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n|$ be the projector onto the GHZ state. Define

$$F_{\mathrm{GHZ}}(\Lambda) := \langle\mathrm{GHZ}_n|\Lambda^{\otimes n}(P_{\mathrm{GHZ}})|\mathrm{GHZ}_n\rangle, \tag{484}$$

the GHZ-ancilla robustness under the local noise Λ, in the same spirit as the general stabilizer robustness functional of Definition 5.17 and the witness framework of [28, 29].

Lemma 8.27 (GHZ robustness in simple noise models)**.** *Let $|\mathrm{GHZ}_n\rangle \in (\mathbb{C}^2)^{\otimes n}$ be as in Definition 8.24, and let $P_{\mathrm{GHZ}} = |\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n|$. Then:*

(a) Under pure Z-type dephasing noise $\Lambda^{(Z)}_{p_Z}(\rho) = (1-p_Z)\rho + p_Z Z\rho Z$ with $0 \le p_Z \le 1$, the GHZ robustness functional of Definition 8.26 is

$$F_{\mathrm{GHZ}}\Big(\Lambda^{(Z)}_{p_Z}\Big) \;=\; \frac{1}{2}\Big(1+\lambda_Z^n\Big), \qquad \lambda_Z := 1-2p_Z. \tag{485}$$

(b) More generally, for any single-qubit Pauli-covariant channel Λ as in Definition 5.11, the GHZ robustness functional $F_{\mathrm{GHZ}}(\Lambda)$ is given by the general stabilizer polynomial of Proposition 5.18 evaluated on the GHZ stabilizer group S_{GHZ}, i.e.

$$F_{\mathrm{GHZ}}(\Lambda) \;=\; \frac{1}{|S_{\mathrm{GHZ}}|}\sum_{g\in S_{\mathrm{GHZ}}} \lambda_X^{w_X(g)}\,\lambda_Y^{w_Y(g)}\,\lambda_Z^{w_Z(g)}, \tag{486}$$

where $\lambda_X, \lambda_Y, \lambda_Z$ are the shrink factors of Λ and the weights $w_P(g)$ are defined as in Definition 5.13. In particular, for any "one-axis dephasing" model (e.g. dephasing along Z or, after a basis rotation, along X), one recovers a closed form of the type (485) with the appropriate shrink factor on the GHZ coherence.[1]

Proof. We first prove part (a) explicitly and then explain the general case in (b).

(a) Pure Z-dephasing. Write

$$|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\Big) = \frac{1}{\sqrt{2}}\Big(|0^n\rangle + |1^n\rangle\Big), \tag{487}$$

---

[1] For a purely X-dephasing model obtained by conjugating $\Lambda^{(Z)}_p$ with Hadamards $H^{\otimes n}$, the same derivation as in part (a) applies in the X-basis, yielding $F_{\mathrm{GHZ}} = \frac{1}{2}(1+\lambda_X^n)$ with $\lambda_X = 1-2p$, consistent with the general stabilizer-based analysis of Proposition 5.18 and standard treatments of Pauli-diagonal noise [6, 5].

so that

$$P_{\mathrm{GHZ}} = |\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n| = \frac{1}{2}\Big(|0^n\rangle\langle 0^n| + |1^n\rangle\langle 1^n| + |0^n\rangle\langle 1^n| + |1^n\rangle\langle 0^n|\Big). \tag{488}$$

The single-qubit dephasing channel $\Lambda^{(Z)}_{p_Z}(\rho) = (1 - p_Z)\rho + p_Z Z\rho Z$ leaves populations in the computational basis invariant and attenuates coherences: for single-qubit operators

$$\Lambda^{(Z)}_{p_Z}\Big(|0\rangle\langle 0|\Big) = |0\rangle\langle 0|\,, \qquad \Lambda^{(Z)}_{p_Z}\Big(|1\rangle\langle 1|\Big) = |1\rangle\langle 1|\,, \tag{489}$$

$$\Lambda^{(Z)}_{p_Z}\Big(|0\rangle\langle 1|\Big) = (1 - 2p_Z)\ |0\rangle\langle 1| = \lambda_Z\ |0\rangle\langle 1|\,, \tag{490}$$

and similarly $\Lambda^{(Z)}_{p_Z}(|1\rangle\langle 0|) = \lambda_Z\ |1\rangle\langle 0|$, where $\lambda_Z := 1 - 2p_Z$ is the Z-shrink factor (cf. the Bloch-picture description of dephasing channels [6, 5]).

By independence across qubits, $\Big(\Lambda^{(Z)}_{p_Z}\Big)^{\otimes n}$ acts as

$$\Lambda^{\otimes n}_{p_Z}\Big(|0^n\rangle\langle 0^n|\Big) = |0^n\rangle\langle 0^n|\,, \tag{491}$$

$$\Lambda^{\otimes n}_{p_Z}\Big(|1^n\rangle\langle 1^n|\Big) = |1^n\rangle\langle 1^n|\,, \tag{492}$$

$$\Lambda^{\otimes n}_{p_Z}\Big(|0^n\rangle\langle 1^n|\Big) = \lambda_Z^n\ |0^n\rangle\langle 1^n|\,, \tag{493}$$

$$\Lambda^{\otimes n}_{p_Z}\Big(|1^n\rangle\langle 0^n|\Big) = \lambda_Z^n\ |1^n\rangle\langle 0^n|\,, \tag{494}$$

since each single-qubit coherence picks up a factor λ_Z, and there are n such factors in $|0^n\rangle\langle 1^n|$.

Hence the noisy GHZ state is

$$\Lambda^{\otimes n}_{p_Z}(P_{\mathrm{GHZ}}) = \frac{1}{2}\Big(|0^n\rangle\langle 0^n| + |1^n\rangle\langle 1^n| + \lambda_Z^n\,|0^n\rangle\langle 1^n| + \lambda_Z^n\,|1^n\rangle\langle 0^n|\Big). \tag{495}$$

By Definition 8.26,

$$F_{\mathrm{GHZ}}\Big(\Lambda^{(Z)}_{p_Z}\Big) = \langle\mathrm{GHZ}_n|\,\Lambda^{\otimes n}_{p_Z}(P_{\mathrm{GHZ}})\,|\mathrm{GHZ}_n\rangle\,. \tag{496}$$

It is convenient to restrict to the two-dimensional subspace $\mathrm{span}\{|0^n\rangle\,, |1^n\rangle\}$. In the ordered basis $\{|0^n\rangle\,, |1^n\rangle\}$, the operator $\Lambda^{\otimes n}_{p_Z}(P_{\mathrm{GHZ}})$ has matrix representation

$$\frac{1}{2}\begin{pmatrix} 1 & \lambda_Z^n \\ \lambda_Z^n & 1 \end{pmatrix}, \tag{497}$$

and the GHZ vector is $|\mathrm{GHZ}_n\rangle = (|0^n\rangle + |1^n\rangle)/\sqrt{2}$, i.e. the column vector $(1,1)^{\mathsf{T}}/\sqrt{2}$ in this basis.

Applying the matrix yields

$$\Lambda^{\otimes n}_{p_Z}(P_{\mathrm{GHZ}})\,|\mathrm{GHZ}_n\rangle = \frac{1}{2}\begin{pmatrix} 1 & \lambda_Z^n \\ \lambda_Z^n & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1 + \lambda_Z^n}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1 + \lambda_Z^n}{2}\,|\mathrm{GHZ}_n\rangle\,. \tag{498}$$

Therefore

$$F_{\mathrm{GHZ}}\Big(\Lambda^{(Z)}_{p_Z}\Big) = \langle\mathrm{GHZ}_n|\,\Lambda^{\otimes n}_{p_Z}(P_{\mathrm{GHZ}})\,|\mathrm{GHZ}_n\rangle = \frac{1 + \lambda_Z^n}{2}, \tag{499}$$

which is (485).

(b) General Pauli-covariant case. Let $S_{\mathrm{GHZ}} \subset \mathcal{P}_n$ denote the GHZ stabilizer group generated by $X^{\otimes n}$ and the pairwise $Z_i Z_{i+1}$ operators as in Definition 8.24. For a stabilizer state, the projector onto its code space is

$$P_{\mathrm{GHZ}} = \frac{1}{|S_{\mathrm{GHZ}}|} \sum_{g \in S_{\mathrm{GHZ}}} g, \tag{500}$$

and this code space is one-dimensional. Thus we may identify $F_{\mathrm{GHZ}}(\Lambda)$ with the general stabilizer robustness functional $F_S(\Lambda)$ of Definition 5.17 for $S = S_{\mathrm{GHZ}}$:

$$F_{\mathrm{GHZ}}(\Lambda) = \mathrm{Tr}\Big[P_{\mathrm{GHZ}}\, \Lambda^{\otimes n}(P_{\mathrm{GHZ}})\Big], \tag{501}$$

or equivalently $F_{\mathrm{GHZ}}(\Lambda) = \langle \mathrm{GHZ}_n |\, \Lambda^{\otimes n}(P_{\mathrm{GHZ}})\, | \mathrm{GHZ}_n \rangle$, as in Definition 8.26.

If Λ is Pauli-covariant with shrink factors $\lambda_X, \lambda_Y, \lambda_Z$ (Definition 5.11), then Proposition 5.18 gives

$$F_{\mathrm{GHZ}}(\Lambda) = F_{S_{\mathrm{GHZ}}}(\Lambda) = \frac{1}{|S_{\mathrm{GHZ}}|} \sum_{g \in S_{\mathrm{GHZ}}} \lambda_X^{w_X(g)}\, \lambda_Y^{w_Y(g)}\, \lambda_Z^{w_Z(g)}, \tag{502}$$

where $w_P(g)$ counts the number of P factors in g for $P \in \{X, Y, Z\}$ (Definition 5.13). This provides a closed-form polynomial expression for F_{GHZ} in terms of the single-site shrink factors, in exact analogy with the surface-code stabilizer robustness polynomial $F_{\mathrm{SC},L}(\Lambda)$ discussed in Section 5.3.1 and Proposition 5.18.

In particular, if the noise is a "one-axis dephasing" channel along some Pauli axis (e.g. the Z-axis treated in part (a), or, after conjugation by Hadamards on each qubit, an X-axis dephasing channel), then only the corresponding GHZ coherence is attenuated by a factor λ^n, and the same projector/2D-subspace calculation as in part (a) yields $F_{\mathrm{GHZ}}(\Lambda) = \frac{1}{2}(1 + \lambda^n)$ for the appropriate shrink factor λ (see, e.g., [6, 5] for the Bloch-sphere description of such channels). This is precisely the situation realized in Assumption 8.25 when the dominant noise on the GHZ ancilla is dephasing along a fixed Pauli direction. □

Remark 8.28. The quantity $1 - F_{\mathrm{GHZ}}$ serves as an effective *ancilla-preparation error probability*. When such ancillas are used as intermediate resources in syndrome extraction or lattice surgery, this error contributes to the effective physical error rate p at the code level, typically with a combinatorial factor reflecting how ancilla faults propagate to data qubits [17, 3, 1, 4, 5].

8.3.2 Example: 3-qubit GHZ ancilla under dephasing

In this example we specialize Definitions 8.24 and 8.26 and Lemma 8.27 to the case $n = 3$ and to a pure dephasing model parametrized by a coherence factor $\mu \in [0, 1]$, following standard treatments of dephasing channels in open-system and QEC analyses [6, 5]. Throughout we abbreviate

$$|\mathrm{GHZ}_3\rangle := \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big), \qquad P_{\mathrm{GHZ}} := |\mathrm{GHZ}_3\rangle\langle \mathrm{GHZ}_3|.$$

Definition 8.29 (Single-qubit dephasing channel)**.** For $\mu \in [0, 1]$ we define the single-qubit dephasing channel

$$\Lambda_\mu(\rho) := E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger, \tag{503}$$

with Kraus operators

$$E_0 := \sqrt{\frac{1+\mu}{2}}\,\mathbb{I}, \qquad E_1 := \sqrt{\frac{1-\mu}{2}}\,Z. \tag{504}$$

This channel leaves computational-basis states invariant and multiplies each single-qubit off-diagonal by the coherence factor μ, as in the standard Bloch-sphere description of dephasing [6].

For the 3-qubit GHZ state of Definition 8.24 we write, as a shorthand,

$$F_{\mathrm{GHZ}}(\mu) := F_{\mathrm{GHZ}}(\Lambda_\mu) = \left\langle \mathrm{GHZ}_3 \middle| \Lambda_\mu^{\otimes 3}(P_{\mathrm{GHZ}}) \middle| \mathrm{GHZ}_3 \right\rangle. \tag{505}$$

Lemma 8.30 (Robustness of GHZ_3 under dephasing)**.** *For the channel Λ_μ of Definition 8.29, the GHZ robustness functional of Definition 8.26 is*

$$F_{\mathrm{GHZ}}(\mu) = \frac{1}{2}\left(1+\mu^3\right). \tag{506}$$

Proof. The dephasing channel Λ_μ coincides with the pure Z-type noise $\Lambda^{(Z)}_{p_Z}$ of Lemma 8.27 with

$$p_Z = \frac{1-\mu}{2}, \qquad \lambda_Z = 1 - 2p_Z = \mu.$$

Specializing Lemma 8.27 to $n = 3$ and $\lambda_Z = \mu$ yields

$$F_{\mathrm{GHZ}}(\mu) = \frac{1}{2}\left(1+\mu^3\right),$$

as claimed. □

Remark 8.31 (Stabilizer-based certification of $F_{\mathrm{GHZ}}(\mu)$)**.** The stabilizer group of GHZ_3 in Definition 8.24 can be generated by

$$Z_1Z_2, \quad Z_2Z_3, \quad X_1X_2X_3.$$

Under pure dephasing, the Z-type stabilizers are unaffected, while

$$\langle X_1X_2X_3\rangle_\mu = \mu^3,$$

where $\langle\cdot\rangle_\mu$ denotes expectation with respect to the noisy state $\Lambda_\mu^{\otimes 3}(P_{\mathrm{GHZ}})$. Combining this with Lemma 8.30 gives

$$F_{\mathrm{GHZ}}(\mu) = \frac{1}{2}\left(1+\mu^3\right) = \frac{1}{2}\left(1+\langle X_1X_2X_3\rangle_\mu\right), \tag{507}$$

so in the dephasing regime $F_{\mathrm{GHZ}}(\mu)$ can be certified by estimating a single global stabilizer expectation value, as is standard in GHZ entanglement experiments and witness-based certification [28, 29].

We now package this robustness into a standard entanglement-witness form, in parallel with the general witness construction of Corollary 5.21. Here the "free" set $\mathcal{F}$ is taken to be the set of biseparable three-qubit states, and the maximal GHZ fidelity over $\mathcal{F}$ is known to be $\alpha_{\max} = \frac{3}{4}$ [28, 29].

Definition 8.32 (GHZ$_3$ entanglement witness)**.** Define the operator

$$W_{\mathrm{GHZ}} := \frac{3}{4}\,\mathbb{I} - P_{\mathrm{GHZ}}, \tag{508}$$

where $P_{\mathrm{GHZ}} = |\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3|$ and $|\mathrm{GHZ}_3\rangle$ is as in Definition 8.24. Then W_{GHZ} is an entanglement witness for genuine tripartite GHZ entanglement: for every biseparable state ρ_{bisep} one has $\mathrm{Tr}(W_{\mathrm{GHZ}}\rho_{\mathrm{bisep}}) \geq 0$, whereas $\mathrm{Tr}\Big(W_{\mathrm{GHZ}}|\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3|\Big) = -\frac{1}{4} < 0$, in line with standard GHZ entanglement-witness constructions [28, 29].

Let

$$\rho_\mu := \Lambda_\mu^{\otimes 3}(P_{\mathrm{GHZ}}) \tag{509}$$

denote the noisy GHZ ancilla under dephasing, with Λ_μ as in Definition 8.29.

Proposition 8.33 (Witness threshold for GHZ$_3$ under dephasing)**.** *For the state ρ_μ defined above, the expectation value of W_{GHZ} is*

$$\langle W_{\mathrm{GHZ}}\rangle_\mu := \mathrm{Tr}\Big(W_{\mathrm{GHZ}}\rho_\mu\Big) = \frac{3}{4} - F_{\mathrm{GHZ}}(\mu) = \frac{1}{4} - \frac{1}{2}\mu^3. \tag{510}$$

In particular, W_{GHZ} detects genuine tripartite entanglement whenever

$$\langle W_{\mathrm{GHZ}}\rangle_\mu < 0 \quad\iff\quad F_{\mathrm{GHZ}}(\mu) > \frac{3}{4} \quad\iff\quad \mu > 2^{-1/3} \approx 0.794, \tag{511}$$

directly realizing the general witness-threshold logic of Corollary 5.21 for this specific GHZ stabilizer.

Proof. We view the GHZ stabilizer as a special case of the general stabilizer framework of Definition 5.17. There, for a stabilizer subgroup S and a single-qubit channel Λ, the robustness functional is

$$F_S(\Lambda) := \mathrm{Tr}\Big[P_S\,\Lambda^{\otimes n}(P_S)\Big], \tag{512}$$

and for any stabilizer state $|\Psi_S\rangle$ with projector $P_S = |\Psi_S\rangle\langle\Psi_S|$ this reduces to

$$F_S(\Lambda) = \langle\Psi_S|\,\Lambda^{\otimes n}(P_S)\,|\Psi_S\rangle\,, \tag{513}$$

cf. Definition 5.17 and eq. (232). In our case S is the three-qubit GHZ stabilizer, $P_S = P_{\mathrm{GHZ}}$, and $F_S(\Lambda_\mu) = F_{\mathrm{GHZ}}(\mu)$ as in Definition 8.26.

The general witness family of Definition 5.19 associates to any real parameter α the stabilizer witness

$$W_\alpha := \alpha\,\mathbb{I} - P_S, \tag{514}$$

and Lemma 5.20 gives, for a noisy stabilizer state $\rho_\Lambda = \Lambda^{\otimes n}(P_S)$,

$$\Big\langle W_\alpha\Big\rangle_\Lambda = \alpha - F_S(\Lambda). \tag{515}$$

Corollary 5.21 then shows that if we choose

$$\alpha_{\max} := \sup_{\sigma\in\mathcal{F}} \langle\Psi_S|\,\sigma\,|\Psi_S\rangle\,, \tag{516}$$

for a convex free set $\mathcal{F}$ (here: the biseparable states), then $W_{\alpha_{\max}}$ is an entanglement / resource witness with detection condition $F_S(\Lambda) > \alpha_{\max}$.

For GHZ$_3$, it is known that the maximal fidelity of $|\mathrm{GHZ}_3\rangle$ with biseparable states is $\alpha_{\max} = 3/4$ [28, 29]. Thus, in our notation,

$$W_{\mathrm{GHZ}} = W_{\alpha_{\max}} = \frac{3}{4}\,\mathbb{I} - P_{\mathrm{GHZ}}, \tag{517}$$

exactly as in Definition 8.32, with free set $\mathcal{F}$ given by the biseparable states.

Now apply Lemma 5.20 with S the GHZ stabilizer, $\Lambda = \Lambda_\mu$, and $\alpha = \alpha_{\max} = 3/4$. For $\rho_\mu = \Lambda_\mu^{\otimes 3}(P_{\mathrm{GHZ}})$ we obtain

$$\langle W_{\mathrm{GHZ}}\rangle_\mu := \mathrm{Tr}\Big(W_{\mathrm{GHZ}}\rho_\mu\Big) = \alpha_{\max} - F_{\mathrm{GHZ}}(\mu) = \frac{3}{4} - F_{\mathrm{GHZ}}(\mu). \tag{518}$$

Using the explicit robustness functional from Lemma 8.30, namely

$$F_{\mathrm{GHZ}}(\mu) = \frac{1}{2}\Big(1 + \mu^3\Big), \tag{519}$$

we find

$$\langle W_{\mathrm{GHZ}}\rangle_\mu = \frac{3}{4} - \frac{1}{2}\Big(1 + \mu^3\Big) \tag{520}$$

$$= \frac{3}{4} - \frac{1}{2} - \frac{1}{2}\mu^3 \tag{521}$$

$$= \frac{1}{4} - \frac{1}{2}\mu^3. \tag{522}$$

This proves the claimed expression for $\langle W_{\mathrm{GHZ}}\rangle_\mu$.

To obtain the detection threshold, we require $\langle W_{\mathrm{GHZ}}\rangle_\mu < 0$, i.e.

$$\frac{1}{4} - \frac{1}{2}\mu^3 < 0 \quad\Longleftrightarrow\quad \mu^3 > \frac{1}{2} \quad\Longleftrightarrow\quad \mu > 2^{-1/3}. \tag{523}$$

In terms of the robustness functional,

$$\langle W_{\mathrm{GHZ}}\rangle_\mu < 0 \quad\Longleftrightarrow\quad \frac{3}{4} - F_{\mathrm{GHZ}}(\mu) < 0 \quad\Longleftrightarrow\quad F_{\mathrm{GHZ}}(\mu) > \frac{3}{4},$$

which is precisely the witness-threshold condition stated in Corollary 5.21, specialized to the GHZ stabilizer with $\alpha_{\max} = 3/4$. This completes the proof. ☐

From the Thesis-valuation point of view, this provides a clean ancilla resource proposition and robustness threshold.

Definition 8.34 (GHZ ancilla proposition)**.** Let φ_{anc} be the resource proposition

$$\varphi_{\mathrm{anc}}: \qquad \text{“The PQS triple realizes a 3-qubit GHZ ancilla.”}$$

For the dephased ancilla state $\rho_\mu := \Lambda_\mu^{\otimes 3}(P_{\mathrm{GHZ}})$, we define its Thesis valuation by

$$v_{\mathrm{Th}}(\rho_\mu; \varphi_{\mathrm{anc}}) := F_{\mathrm{GHZ}}(\mu), \tag{524}$$

where $F_{\mathrm{GHZ}}(\mu)$ is the robustness functional of Definition 8.26 and Lemma 8.30. The entanglement witness W_{GHZ} of Definition 8.32 and the threshold condition of Proposition 8.33 provide a hard certification threshold: whenever $\mu > 2^{-1/3}$ (equivalently $F_{\mathrm{GHZ}}(\mu) > 3/4$), ρ_μ is guaranteed to be genuinely tripartite entangled, in analogy with the general stabilizer-witness thresholds of Corollary 5.21.

Remark 8.35 (Ancilla robustness in a logical error budget)**.** Suppose a repetition-type QEC gadget is used whose ideal logical error probability (with perfect ancillas) is $p_L(p)$ as a function of the physical bit-flip rate p. For the three-qubit repetition code of Proposition 8.7, this is

$$p_L^{(\mathrm{rep})}(p) = 3p^2 - 2p^3.$$

Now assume that each QEC round also uses a GHZ_3 ancilla with dephasing parameter μ, prepared in the IQR arena and then consumed in the gadget. Writing $F_{\mathrm{GHZ}}(\mu)$ for the corresponding ancilla robustness, we can express to leading order a schematic error budget of the form

$$p_L^{(\mathrm{real})}(p,\mu) \;\approx\; p_L(p) + \beta\Big(1 - F_{\mathrm{GHZ}}(\mu)\Big) + \dots, \tag{525}$$

where $\beta \geq 0$ is a circuit-dependent combinatorial factor counting how ancilla faults propagate to logical errors in the specific syndrome-extraction or lattice-surgery construction [17, 3, 1, 5]. The ellipsis denotes higher-order terms in p and $1 - F_{\mathrm{GHZ}}(\mu)$, such as joint events where both data and ancilla faults occur in the same round.

Equivalently, denoting by $\rho'_{\log}$ the actual noisy logical output state, the logical Thesis valuation for the correctness proposition φ_L (cf. Definition 8.8 and Corollary 8.9) can be written, to the same order, as

$$\upsilon_{\mathrm{Th}}(\rho'_{\log};\varphi_L) \approx 1 - p_L(p) - \beta\Big(1 - F_{\mathrm{GHZ}}(\mu)\Big). \tag{526}$$

This makes explicit how ancilla quality $F_{\mathrm{GHZ}}(\mu)$ and code robustness $p_L(p)$ enter on an equal footing in the logical error budget: a degradation of the GHZ ancilla (smaller $F_{\mathrm{GHZ}}(\mu)$, i.e. smaller $\upsilon_{\mathrm{Th}}(\rho_\mu;\varphi_{\mathrm{anc}})$) feeds directly into an increased logical failure rate. In the full surface-code setting, such ancilla imperfections renormalize the effective physical parameters $(p, \delta, \varepsilon_{\mathrm{ps}})$ entering the IQR threshold theorem (Theorem 7.2), exactly as in standard ancilla-based fault-tolerant architectures [17, 3, 1, 5].

8.3.3 Folding GHZ noise into $(p, \delta, \varepsilon_{\mathrm{ps}})$

Assume a given QEC scheme uses N_{anc} GHZ states per cycle, each of size n, and that each ancilla is consumed via a circuit that can propagate a single ancilla fault into at most w_{eff} additional data-qubit errors, as in standard ancilla-assisted fault-tolerant constructions and error-budget estimates [17, 3, 1, 5]. Let n_L be the number of data qubits in the code block (e.g. $n_L = n_L(L)$ for a surface-code patch at size L).

Lemma 8.36 (Effective contribution of GHZ noise to p)**.** *Let ρ_{anc} be the (possibly noisy) GHZ ancilla state used in a single QEC cycle, and let $P_{\mathrm{GHZ}} = |\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n|$ be the ideal projector. Define*

$$F_{\mathrm{GHZ}}(\Lambda) := \mathrm{Tr}\Big(P_{\mathrm{GHZ}}\,\rho_{\mathrm{anc}}\Big), \tag{527}$$

as in Definition 8.26. Let p_{GHZ} *denote the effective probability that the ancilla is* faulty *in the relevant stabilizer sector, i.e. that it lies outside the ideal GHZ state component. Then*

$$p_{\mathrm{GHZ}} \;\le\; 1 - F_{\mathrm{GHZ}}(\Lambda). \tag{528}$$

If each faulty ancilla can cause at most w_{eff} *additional data-qubit errors, and* N_{anc} *ancillas are used per cycle, then the ancilla-induced increment to the* average *per-data-qubit error rate per cycle satisfies*

$$\Delta p_{\mathrm{anc}} \;\lesssim\; \frac{N_{\mathrm{anc}}\, w_{\mathrm{eff}}}{n_L}\, p_{\mathrm{GHZ}}. \tag{529}$$

Equivalently, up to a code-architecture dependent constant $\kappa_{\mathrm{anc}} := N_{\mathrm{anc}} w_{\mathrm{eff}}/n_L = O(1)$*, we can write*

$$\Delta p_{\mathrm{anc}} \;\lesssim\; \kappa_{\mathrm{anc}}\, p_{\mathrm{GHZ}}, \tag{530}$$

and the total effective per-PQS error rate entering Theorem 7.2 is

$$p_{\mathrm{total}} \;\approx\; p_{\mathrm{data}} + \Delta p_{\mathrm{anc}}, \tag{531}$$

where p_{data} *is the contribution from direct data-qubit noise induced by* $\Delta\hat{G}$*, in analogy with standard QEC error-budget decompositions [17, 3, 5].*

Proof. By definition of $F_{\mathrm{GHZ}}(\Lambda)$, we can decompose the ancilla state as

$$\rho_{\mathrm{anc}} = F_{\mathrm{GHZ}}(\Lambda)\, P_{\mathrm{GHZ}} + \Big(1 - F_{\mathrm{GHZ}}(\Lambda)\Big)\, \sigma_{\perp}, \tag{532}$$

where $\sigma_{\perp}$ is a density operator supported on the subspace orthogonal to $|\mathrm{GHZ}_n\rangle$ in the relevant stabilizer sector (for example, the orthogonal stabilizer eigenspaces appearing in the GHZ stabilizer group). Interpreting $F_{\mathrm{GHZ}}(\Lambda)$ as the population in the ideal GHZ sector, the weight

$$1 - F_{\mathrm{GHZ}}(\Lambda) \tag{533}$$

is the total probability mass in sectors that differ from the ideal GHZ state. If p_{GHZ} is defined as the effective probability that the ancilla is faulty in precisely those sectors that can induce errors in the gadget (a possibly more restrictive notion than "any orthogonal component"), then by construction

$$p_{\mathrm{GHZ}} \;\le\; 1 - F_{\mathrm{GHZ}}(\Lambda), \tag{534}$$

with equality in the worst-case modelling where *all* orthogonal components are treated as faults.

Now consider one QEC cycle. Each ancilla is independently either good or faulty (in this effective model). When a given ancilla is faulty, the gadget assumption says that at most w_{eff} data-qubit errors can be induced by that ancilla in that cycle. Thus, if we denote by $N_{\mathrm{err}}^{(\mathrm{anc})}$ the random number of ancilla-induced data errors in one cycle, we have the bound

$$\mathbb{E}\Big[N_{\mathrm{err}}^{(\mathrm{anc})}\Big] \;\le\; N_{\mathrm{anc}}\, w_{\mathrm{eff}}\, p_{\mathrm{GHZ}}, \tag{535}$$

since each of the N_{anc} ancillas contributes at most w_{eff} errors with probability at most p_{GHZ}, and we use the union/linearity of expectation argument exactly as in standard fault-tolerant counting estimates [17, 3, 5].

Dividing by n_L (the number of data qubits) yields an upper bound on the average additional error probability per data qubit:

$$\Delta p_{\mathrm{anc}} := \frac{1}{n_L}\,\mathbb{E}\Big[N_{\mathrm{err}}^{(\mathrm{anc})}\Big] \;\le\; \frac{N_{\mathrm{anc}}\, w_{\mathrm{eff}}}{n_L}\, p_{\mathrm{GHZ}}. \tag{536}$$

For a fixed code architecture, the ratio $\kappa_{\mathrm{anc}} := N_{\mathrm{anc}} w_{\mathrm{eff}}/n_L$ is a constant independent of L (e.g. bounded by a small number for local, bounded-width gadgets), so we may absorb it into the implicit constant in the "$\lesssim$" notation and write

$$\Delta p_{\mathrm{anc}} \;\lesssim\; \kappa_{\mathrm{anc}}\, p_{\mathrm{GHZ}}. \tag{537}$$

Adding this contribution to the base per-qubit rate p_{data} from direct data noise yields the effective p_{total} entering the local-Pauli description of Theorem 7.2. $\square$

Similarly, differences in ancilla-preparation noise between schedules A and B contribute to the process-differential parameter δ in the holosymmetric comparison.

Lemma 8.37 (Ancilla contribution to δ)**.** *Let $\Lambda_{\mathrm{GHZ}}^{(A)}$ and $\Lambda_{\mathrm{GHZ}}^{(B)}$ be the ancilla preparation channels for schedules A and B, respectively, and denote the resulting ancilla states by*

$$\rho_{\mathrm{anc}}^{(A)} := \Lambda_{\mathrm{GHZ}}^{(A)}(P_{\mathrm{GHZ}}), \qquad \rho_{\mathrm{anc}}^{(B)} := \Lambda_{\mathrm{GHZ}}^{(B)}(P_{\mathrm{GHZ}}).$$

Let $q_{\mathrm{GHZ}}^{(S)}(\vec{P})$ be the Pauli distribution of $\Lambda_{\mathrm{GHZ}}^{(S)}$ on the n-qubit ancilla, for $S \in \{A, B\}$ and $\vec{P}$ an n-fold Pauli string. The dual-history GHZ overlap for a single ancilla is

$$C_{\mathrm{GHZ}} := \sum_{\vec{P}} \sqrt{q_{\mathrm{GHZ}}^{(A)}(\vec{P})\, q_{\mathrm{GHZ}}^{(B)}(\vec{P})}, \tag{538}$$

in direct analogy with the Pauli-overlap structure of Theorem 6.12. Define

$$\delta_{\mathrm{GHZ}} := 1 - C_{\mathrm{GHZ}}. \tag{539}$$

If N_{anc} such ancillas are used per cycle, independently prepared in each schedule, then the ancilla-induced contribution to the total process differential at the code level satisfies

$$\Delta\delta_{\mathrm{anc}} \;\lesssim\; N_{\mathrm{anc}}\, \delta_{\mathrm{GHZ}}, \tag{540}$$

so that, up to architecture-dependent constants,

$$\delta_{\mathrm{total}} \approx \delta_{\mathrm{data}} + \Delta\delta_{\mathrm{anc}}, \tag{541}$$

where δ_{data} is the contribution from differences in the data-level channels.

Proof. By Theorem 6.12, the holosymmetric overlap associated with a Pauli channel is given by a product of single-site overlaps over the PQS touched by the gadget. For a GHZ ancilla treated as an n-qubit block, we may formally collapse its Pauli statistics into an *effective*

Pauli distribution $q^{(S)}_{\mathrm{GHZ}}(\vec{P})$ over n-fold strings $\vec{P}$, and the corresponding dual-history overlap factor for that block is precisely

$$C_{\mathrm{GHZ}} = \sum_{\vec{P}} \sqrt{q^{(A)}_{\mathrm{GHZ}}(\vec{P})\, q^{(B)}_{\mathrm{GHZ}}(\vec{P})}, \tag{542}$$

the natural generalization of the single-site overlap of Definition 6.11 to an n-qubit resource.

If one QEC cycle uses N_{anc} such ancillas, and we assume the ancillas are prepared independently and used in a way that factorizes at the level of Pauli statistics (the standard assumption in circuit-level ancilla modelling), then the total ancilla-induced overlap factor in the dual-history interferometer for that cycle is the product of the overlaps of the individual ancillas:

$$C_{\mathrm{anc,tot}} = C_{\mathrm{GHZ}}^{N_{\mathrm{anc}}} = (1-\delta_{\mathrm{GHZ}})^{N_{\mathrm{anc}}}. \tag{543}$$

The corresponding ancilla-induced differential is

$$\Delta\delta_{\mathrm{anc}} := 1 - C_{\mathrm{anc,tot}} = 1 - (1-\delta_{\mathrm{GHZ}})^{N_{\mathrm{anc}}}. \tag{544}$$

For $0 \le \delta_{\mathrm{GHZ}} \le 1$, we have the standard inequality

$$1 - (1-\delta_{\mathrm{GHZ}})^{N_{\mathrm{anc}}} \;\le\; N_{\mathrm{anc}}\, \delta_{\mathrm{GHZ}}, \tag{545}$$

which follows, for example, from Bernoulli's inequality or the binomial expansion with all higher-order terms nonnegative. Thus

$$\Delta\delta_{\mathrm{anc}} \;\le\; N_{\mathrm{anc}}\, \delta_{\mathrm{GHZ}}. \tag{546}$$

Up to architecture-dependent constants (e.g. absorbing any additional ancilla-related contributions into the implicit constant in "$\lesssim$"), this yields the claimed scaling

$$\Delta\delta_{\mathrm{anc}} \;\lesssim\; N_{\mathrm{anc}}\, \delta_{\mathrm{GHZ}}. \tag{547}$$

Since the total process differential at the code level arises from both data-channel differences and ancilla-channel differences, it is natural to write, at leading order,

$$\delta_{\mathrm{total}} \approx \delta_{\mathrm{data}} + \Delta\delta_{\mathrm{anc}}, \tag{548}$$

consistent with the way ancilla-induced differences are folded into the holosymmetric process threshold in Theorems 6.21 and 7.2. ☐

Remark 8.38. A similar analysis applies to cluster-state ancillas (linear chains or small 2D clusters) used in measurement-based or lattice-surgery style gadgets [30, 31, 32, 11, 12]. Each such resource state has its own robustness polynomial $F_{\mathrm{cluster}}(\Lambda)$ (in the sense of Definition 5.17) and its own Pauli-overlap factor C_{cluster}, which contribute to p_{total} and δ_{total} exactly as in the GHZ case:

$$\Delta p_{\mathrm{anc,cluster}} \;\lesssim\; \kappa_{\mathrm{anc,cluster}}\Big(1 - F_{\mathrm{cluster}}(\Lambda)\Big), \tag{549}$$

$$\Delta\delta_{\mathrm{anc,cluster}} \;\lesssim\; N_{\mathrm{anc,cluster}}\, \delta_{\mathrm{cluster}}, \tag{550}$$

with architecture-dependent constants $\kappa_{\mathrm{anc,cluster}}$ and $N_{\mathrm{anc,cluster}}$. In all cases, once the resource-state robustness F_{resource} and process overlap C_{resource} are expressed in terms of local Pauli parameters, their effects can be *absorbed* into the effective $(p, \delta, \varepsilon_{\mathrm{ps}})$ entering the global IQR threshold theorem (Theorem 7.2), mirroring the standard treatment of noisy ancillas in fault-tolerant architectures [17, 3, 1, 5].

8.3.4 Example: 4-qubit cluster ancilla under depolarising noise

We now give a concrete instance of the general stabilizer robustness and holosymmetry framework for a small cluster-state ancilla, using the standard 4-qubit linear cluster [31, 32].

Definition 8.39 (4-qubit linear cluster state)**.** On a four-qubit register $\mathcal{H}_{\mathrm{cl}} := (\mathbb{C}^2)^{\otimes 4}$, the (open-chain) 4-qubit cluster state is the graph state associated with the path graph on vertices $\{1, 2, 3, 4\}$:

$$|C_4\rangle := \Big(\prod_{(i,j)\in E} \mathrm{CZ}_{ij}\Big)|+\rangle^{\otimes 4}, \tag{551}$$

where $E = \{(1,2),(2,3),(3,4)\}$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, as in the standard graph-state/cluster-state formalism [31, 32, 30]. Its stabilizer group $S_{C_4} \subset \mathcal{P}_4$ is generated by

$$K_1 = X_1Z_2, \quad K_2 = Z_1X_2Z_3, \quad K_3 = Z_2X_3Z_4, \quad K_4 = Z_3X_4, \tag{552}$$

in accordance with the usual graph-state stabilizer rules [32, 13]. We write $P_{C_4} := |C_4\rangle\langle C_4|$ for the projector onto the cluster state.

Definition 8.40 (Isotropic single-qubit depolarising channel)**.** For a parameter $\eta \in [-1, 1]$ we define the isotropic single-qubit depolarising channel

$$\Lambda_\eta(\rho) := \eta\,\rho + (1-\eta)\,\frac{\mathbb{I}}{2}\,\mathrm{Tr}(\rho). \tag{553}$$

In the Pauli basis this satisfies

$$\Lambda_\eta(\mathbb{I}) = \mathbb{I}, \qquad \Lambda_\eta(P) = \eta\,P \quad \text{for } P \in \{X, Y, Z\}, \tag{554}$$

so Λ_η is Pauli-covariant with shrink factors $\lambda_X = \lambda_Y = \lambda_Z = \eta$ in the sense of Definition 5.11, coinciding with the standard textbook depolarising channel [6].

For the 4-qubit cluster state we define, in the notation of Definition 5.17,

$$F_{C_4}(\eta) := F_{S_{C_4}}(\Lambda_\eta) = \Big\langle C_4\Big|\Lambda_\eta^{\otimes 4}(P_{C_4})\Big|C_4\Big\rangle. \tag{555}$$

Lemma 8.41 (Robustness of a 4-qubit cluster under depolarisation)**.** *For the 4-qubit cluster state of Definition 8.39 and the depolarising channel Λ_η of Definition 8.40, the stabilizer robustness functional is*

$$F_{C_4}(\eta) = \frac{1}{16}\Big(1 + 2\eta^2 + 8\eta^3 + 5\eta^4\Big). \tag{556}$$

Proof. By Definition 5.17 and Proposition 5.18, for a Pauli-covariant single-qubit channel with shrink factors $\lambda_X, \lambda_Y, \lambda_Z$ the stabilizer robustness for a stabilizer *state* with group S is

$$F_S(\Lambda) = \frac{1}{|S|}\sum_{g\in S} \lambda_X^{w_X(g)}\lambda_Y^{w_Y(g)}\lambda_Z^{w_Z(g)}, \tag{557}$$

where $w_P(g)$ counts the number of tensor factors equal to $P \in \{X, Y, Z\}$. In the isotropic depolarising case of Definition 8.40 we have

$$\lambda_X = \lambda_Y = \lambda_Z = \eta,$$

so for $g \in S_{C_4}$ with total Pauli weight $w(g) := w_X(g) + w_Y(g) + w_Z(g)$ we simply obtain

$$F_{S_{C_4}}(\Lambda_\eta) = \frac{1}{|S_{C_4}|} \sum_{g \in S_{C_4}} \eta^{w(g)}. \tag{558}$$

Thus it suffices to determine the weight distribution of the 4-qubit cluster stabilizer group S_{C_4}.

Step 1: Structure of S_{C_4}. The stabilizer generators given in eq. (552) are

$$K_1 = X_1 Z_2 I_3 I_4, \quad K_2 = Z_1 X_2 Z_3 I_4, \quad K_3 = I_1 Z_2 X_3 Z_4, \quad K_4 = I_1 I_2 Z_3 X_4. \tag{559}$$

These are four independent, commuting Pauli operators, so the stabilizer group they generate has $|S_{C_4}| = 2^4 = 16$ elements, as usual for a 4-qubit stabilizer state [13, 32].

We ignore overall ±1 phases, since the robustness polynomial depends only on the underlying Pauli operator, not on the global sign. Every $g \in S_{C_4}$ is a product of a subset of the K_i, and we write $w(g)$ for the number of non-identity single-qubit factors of g.

Step 2: Explicit weight counting. We now enumerate the 16 elements and their weights.

- Identity:

$$\mathbb{I} := I_1 I_2 I_3 I_4, \qquad w(\mathbb{I}) = 0.$$

- Single generators:

$$K_1 = X_1 Z_2 I_3 I_4, \qquad w(K_1) = 2, \tag{560}$$
$$K_2 = Z_1 X_2 Z_3 I_4, \qquad w(K_2) = 3, \tag{561}$$
$$K_3 = I_1 Z_2 X_3 Z_4, \qquad w(K_3) = 3, \tag{562}$$
$$K_4 = I_1 I_2 Z_3 X_4, \qquad w(K_4) = 2. \tag{563}$$

  Thus among the 4 single-generator elements we have two of weight 2 and two of weight 3.

- Products of two generators (we use the Pauli multiplication rules, ignoring global phases):

$$K_1K_2 = (X_1Z_2)(Z_1X_2Z_3) = Y_1Y_2Z_3I_4, \qquad w(K_1K_2) = 3, \tag{564}$$
$$K_1K_3 = (X_1Z_2)(Z_2X_3Z_4) = X_1I_2X_3Z_4, \qquad w(K_1K_3) = 3, \tag{565}$$
$$K_1K_4 = (X_1Z_2)(Z_3X_4) = X_1Z_2Z_3X_4, \qquad w(K_1K_4) = 4, \tag{566}$$
$$K_2K_3 = (Z_1X_2Z_3)(Z_2X_3Z_4) = Z_1Y_2Y_3Z_4, \qquad w(K_2K_3) = 4, \tag{567}$$
$$K_2K_4 = (Z_1X_2Z_3)(Z_3X_4) = Z_1X_2I_3X_4, \qquad w(K_2K_4) = 3, \tag{568}$$
$$K_3K_4 = (Z_2X_3Z_4)(Z_3X_4) = I_1Z_2Y_3Y_4, \qquad w(K_3K_4) = 3. \tag{569}$$

  Hence among the 6 pairwise products we get 4 elements of weight 3 and 2 elements of weight 4.

- Products of three generators:

$$K_1K_2K_3 = (K_1K_2)K_3 = (Y_1Y_2Z_3I_4)(Z_2X_3Z_4) = Y_1X_2Y_3Z_4, \quad w(K_1K_2K_3) = 4, \tag{570}$$

$$K_1K_2K_4 = (K_1K_2)K_4 = (Y_1Y_2Z_3I_4)(Z_3X_4) = Y_1Y_2I_3X_4, \quad w(K_1K_2K_4) = 3, \tag{571}$$

$$K_1K_3K_4 = (K_1K_3)K_4 = (X_1I_2X_3Z_4)(Z_3X_4) = X_1I_2Y_3Y_4, \quad w(K_1K_3K_4) = 3, \tag{572}$$

$$K_2K_3K_4 = (K_2K_3)K_4 = (Z_1Y_2Y_3Z_4)(Z_3X_4) = Z_1Y_2X_3Y_4, \quad w(K_2K_3K_4) = 4. \tag{573}$$

  Thus among the 4 triple products there are 2 of weight 3 and 2 of weight 4.

- Product of all four generators:

$$\begin{aligned} K_1K_2K_3K_4 &= (K_1K_2K_3)K_4 \\ &= (Y_1X_2Y_3Z_4)(Z_3X_4) \\ &= Y_1X_2X_3Y_4, \quad \text{with } w(K_1K_2K_3K_4) = 4. \end{aligned}$$

Collecting all multiplicities, we obtain the weight distribution in S_{C_4}:

$$\begin{aligned} &\#\{g \in S_{C_4} : w(g) = 0\} = 1, \\ &\#\{g \in S_{C_4} : w(g) = 2\} = 2, \\ &\#\{g \in S_{C_4} : w(g) = 3\} = 2 \text{ (singles)} + 4 \text{ (pairs)} + 2 \text{ (triples)} = 8, \\ &\#\{g \in S_{C_4} : w(g) = 4\} = 2 \text{ (pairs)} + 2 \text{ (triples)} + 1 \text{ (quadruple)} = 5. \end{aligned}$$

As a sanity check, $1 + 2 + 8 + 5 = 16 = |S_{C_4}|$.

Step 3: Plug into the robustness polynomial. Using the weight distribution in the stabilizer-polynomial expression, we find

$$F_{C_4}(\eta) = \frac{1}{16} \sum_{g \in S_{C_4}} \eta^{w(g)} \tag{574}$$

$$= \frac{1}{16}\Big(1 \cdot \eta^0 + 2 \cdot \eta^2 + 8 \cdot \eta^3 + 5 \cdot \eta^4\Big), \tag{575}$$

which is precisely eq. (556). This matches the general stabilizer robustness structure for graph states under isotropic depolarisation [32, 6] and satisfies the expected limits $F_{C_4}(1) = 1$ (no noise) and $F_{C_4}(0) = 1/16$ (completely mixed output). □

Remark 8.42. The functional $F_{C_4}(\eta)$ in eq. (556) provides the Thesis valuation for the cluster ancilla proposition

$$\varphi_{C_4} : \quad \text{"The PQS chain realizes the 4-qubit cluster resource } |C_4\rangle\text{."}$$

in the isotropically depolarised state $\rho_\eta = \Lambda_\eta^{\otimes 4}(P_{C_4})$: $v_{\mathrm{Th}}(\rho_\eta; \varphi_{C_4}) = F_{C_4}(\eta)$. This is a direct instance of the general stabilizer robustness framework of Definition 5.17 and Proposition 5.18.

We now specialise the Pauli-channel holosymmetry theorem of Theorem 6.12 to the case of depolarising noise on cluster ancillas.

Definition 8.43 (Depolarising family for cluster ancillas)**.** Let $\Lambda^{(A)}_{p_A}$ and $\Lambda^{(B)}_{p_B}$ be two single-qubit depolarising channels acting on a cluster ancilla in schedules A and B, respectively, with Kraus decompositions

$$\Lambda^{(\nu)}_{p_\nu}(\rho) = \sum_{r=0}^{3} E^{(\nu)}_r \rho \, E^{(\nu)\dagger}_r, \qquad \nu \in \{A, B\}, \tag{576}$$

where

$$E^{(\nu)}_0 := \sqrt{p_\nu}\, \mathbb{I}, \qquad E^{(\nu)}_j := \sqrt{\frac{1-p_\nu}{3}}\, P_j, \quad P_1 = X,\ P_2 = Y,\ P_3 = Z. \tag{577}$$

Here $p_\nu \in [0,1]$ is the "no-error" probability, and $(1-p_\nu)$ is the total probability of a nontrivial Pauli error on the cluster qubit, as in the canonical depolarising model [6].

Definition 8.44 (Depolarising Pauli-overlap function)**.** For $p_A, p_B \in [0,1]$ we define

$$k_{\mathrm{dep}}(p_A, p_B) := \sqrt{p_A p_B} + \sqrt{(1-p_A)(1-p_B)}. \tag{578}$$

Lemma 8.45 (Single-site Pauli overlap for depolarising noise)**.** *Let* $\Lambda^{(A)}_{p_A}$ *and* $\Lambda^{(B)}_{p_B}$ *be as in Definition 8.43, and let* $\mathbf{q}^{(A)}$*,* $\mathbf{q}^{(B)}$ *be the corresponding Pauli error distributions at a given site in schedules* A *and* B*. Then the single-site Pauli overlap of Definition 6.11 is*

$$k_x = \sum_{P \in \{\mathbb{I},X,Y,Z\}} \sqrt{q^{(A)}(P)\, q^{(B)}(P)} = k_{\mathrm{dep}}(p_A, p_B), \tag{579}$$

where k_{dep} *is the depolarising overlap function of Definition 8.44.*

Proof. By Definition 8.43, a single-qubit depolarising channel $\Lambda^{(\nu)}_{p_\nu}$ has Kraus operators

$$E^{(\nu)}_0 = \sqrt{p_\nu}\, \mathbb{I}, \qquad E^{(\nu)}_j = \sqrt{\frac{1-p_\nu}{3}}\, P_j, \quad P_1 = X,\ P_2 = Y,\ P_3 = Z, \tag{580}$$

for $\nu \in \{A, B\}$. This is precisely a Pauli channel with error distribution

$$q^{(\nu)}(\mathbb{I}) = p_\nu, \qquad q^{(\nu)}(X) = q^{(\nu)}(Y) = q^{(\nu)}(Z) = \frac{1-p_\nu}{3}, \tag{581}$$

since $\Lambda^{(\nu)}_{p_\nu}$ applies $\mathbb{I}$ with probability p_ν and each nontrivial Pauli with probability $(1-p_\nu)/3$, as in the canonical single-qubit depolarising model [6].

By Definition 6.11, the single-site Pauli overlap at a given site x is

$$k_x = \sum_{P \in \{\mathbb{I},X,Y,Z\}} \sqrt{q^{(A)}(P)\, q^{(B)}(P)}. \tag{582}$$

Substituting the depolarising distributions, we obtain

$$\begin{aligned}
k_x &= \sqrt{q^{(A)}(\mathbb{I})\, q^{(B)}(\mathbb{I})} + \sum_{P\in\{X,Y,Z\}} \sqrt{q^{(A)}(P)\, q^{(B)}(P)} \\
&= \sqrt{p_A p_B} + \sum_{P\in\{X,Y,Z\}} \sqrt{\frac{1-p_A}{3}\cdot\frac{1-p_B}{3}} \\
&= \sqrt{p_A p_B} + 3\cdot\frac{1}{3}\sqrt{(1-p_A)(1-p_B)} \\
&= \sqrt{p_A p_B} + \sqrt{(1-p_A)(1-p_B)}. \qquad (583)
\end{aligned}$$

By Definition 8.44, the right-hand side is exactly $k_{\text{dep}}(p_A, p_B)$, so

$$k_x = k_{\text{dep}}(p_A, p_B), \qquad (584)$$

as claimed.

As a sanity check, if $p_A = p_B$ then $k_x = \sqrt{p_A^2} + \sqrt{(1-p_A)^2} = 1$, corresponding to identical Pauli distributions (no process distinguishability at that site), and if $p_A = 1$, $p_B = 0$ then $k_x = 0$, corresponding to orthogonal Pauli distributions. Both behaviours are consistent with the interpretation of k_x as a single-site overlap parameter. □

Lemma 8.46 (Holosymmetry of a 4-qubit cluster under depolarising noise)**.** *Consider a 4-qubit cluster ancilla $|C_4\rangle$ as in Definition 8.39, used inside a gadget where four PQS loci carry independent depolarising noise with parameters p_A, p_B in schedules A and B, respectively. Let Q_{C_4} denote the set of the four qubits in the cluster. Then the dual-history cluster overlap is*

$$C^{(4)}_{\text{cluster}}(p_A, p_B) = \prod_{x\in Q_{C_4}} k_x = \Big(k_{\text{dep}}(p_A, p_B)\Big)^4, \qquad (585)$$

and the corresponding holosymmetry valuation for the cluster-resource process proposition is

$$v^{(C_4)}_{\text{Holo}}(p_A, p_B) = \frac{1}{2}\bigg(1 + \Big(k_{\text{dep}}(p_A, p_B)\Big)^4\bigg), \qquad (586)$$

consistent with the general Pauli-channel holosymmetry factorisation of Theorem 6.12.

Proof. We first recall the structure of the dual-history interferometer in the Pauli-channel regime. For a given gadget and a fixed input state on the system (here including the cluster ancilla), the dual-history construction of Definitions 6.3 and 6.4 uses a reference qubit R prepared in $|+\rangle_R$, applies a controlled channel that applies schedule A when $R = 0$ and schedule B when $R = 1$, and finally applies a Hadamard H_R followed by a measurement of $|0\rangle\langle 0|_R$. The holosymmetric overlap C is then obtained from the off-diagonal coherence between the two histories in the $\{|0\rangle_R, |1\rangle_R\}$ basis:

$$v^{(L)}_{\text{Holo}} = \frac{1}{2}(1 + C^{(L)}).$$

Under Assumption 6.10 (independent single-qubit Pauli noise per PQS, possibly with different distributions for A and B), Theorem 6.12 shows that this overlap factorises as

$$C^{(L)} = \prod_{x\in Q} k_x, \tag{587}$$

where Q is the set of PQS touched by the gadget and k_x is the single-site Pauli overlap at x,

$$k_x := \sum_{P\in\{\mathbb{I},X,Y,Z\}} \sqrt{q_x^{(A)}(P)\, q_x^{(B)}(P)}, \tag{588}$$

with $q_x^{(A)}$ and $q_x^{(B)}$ the Pauli error distributions for schedules A and B at that site (cf. Definition 6.11).

In the present setting, the gadget under consideration uses a 4-qubit cluster ancilla on the sites Q_{C_4}, and we assume that each of these four PQS loci carries independent single-qubit depolarising noise, with parameter p_A for schedule A and p_B for schedule B, as in Definition 8.43. Thus at each cluster site $x \in Q_{C_4}$, the corresponding Pauli error distributions $\mathbf{q}^{(A)}$ and $\mathbf{q}^{(B)}$ are exactly those of the depolarising channels $\Lambda_{p_A}^{(A)}$ and $\Lambda_{p_B}^{(B)}$.

By Lemma 8.45, the single-site overlap at any such site is

$$k_x = k_{\text{dep}}(p_A, p_B) := \sqrt{p_A p_B} + \sqrt{(1-p_A)(1-p_B)}. \tag{589}$$

Since the noise model is homogeneous across the four cluster qubits, the same overlap value $k_{\text{dep}}(p_A, p_B)$ applies at each $x \in Q_{C_4}$.

Applying the factorisation of Theorem 6.12 to the set Q_{C_4} (with $|Q_{C_4}| = 4$) therefore yields

$$C_{\text{cluster}}^{(4)}(p_A, p_B) = \prod_{x\in Q_{C_4}} k_x = \Big(k_{\text{dep}}(p_A, p_B)\Big)^4. \tag{590}$$

Finally, the holosymmetric process valuation for this 4-qubit cluster resource comparison is obtained from the general relation $v_{\text{Holo}} = \frac{1}{2}(1 + C)$ (cf. Definition 6.4 and eq. (331)), giving

$$v_{\text{Holo}}^{(C_4)}(p_A, p_B) = \frac{1}{2}\Big(1 + C_{\text{cluster}}^{(4)}(p_A, p_B)\Big) = \frac{1}{2}\Big(1 + \Big(k_{\text{dep}}(p_A, p_B)\Big)^4\Big), \tag{591}$$

as claimed. As a sanity check:

- If $p_A = p_B$, then $k_{\text{dep}}(p_A, p_A) = 1$, so $C_{\text{cluster}}^{(4)} = 1$ and $v_{\text{Holo}}^{(C_4)} = 1$, corresponding to perfectly matched ancilla noise in the two schedules.
- If $p_A = 1$ and $p_B = 0$ (one schedule always applies $\mathbb{I}$ and the other always applies a nontrivial Pauli), then $k_{\text{dep}}(1, 0) = 0$, so $C_{\text{cluster}}^{(4)} = 0$ and $v_{\text{Holo}}^{(C_4)} = \frac{1}{2}$, i.e. fully incoherent, maximally distinguishable ancilla processes at that level.

These limiting cases match the interpretation of k_{dep} as a single-site classical overlap and of v_{Holo} as an interferometric process-visibility in the IQR setting. □

Remark 8.47. The robustness polynomial $F_{C_4}(\eta)$ of Lemma 8.41 and the depolarising overlap $k_{\text{dep}}(p_A, p_B)$ of Definition 8.44 and Lemma 8.45 together give a concrete cluster-ancilla instance of the abstract quantities $F_{\text{cluster}}(\Lambda)$ and C_{cluster} mentioned earlier, within the general stabilizer and witness framework of Section 5.3.1 and [13, 28, 29].

In particular:

- $F_{C_4}(\eta)$ controls the *state-level* quality of a single noisy cluster ancilla under depolarisation, and so enters the effective physical error rate p and leakage/error budget along the same lines as F_{GHZ} in Lemmas 8.36 and 8.37.
- $k_{\mathrm{dep}}(p_A, p_B)$ controls the *process-level* similarity of ancilla noise between two schedules and thus feeds directly into the ancilla-induced contribution to the global differential parameter δ via factors of the form $\left(k_{\mathrm{dep}}(p_A, p_B)\right)^{|Q_{C_4}|}$, exactly as in Lemma 8.37.

Once these cluster-ancilla quantities are expressed in terms of local Pauli parameters (η) and (p_A, p_B), their contributions can be folded into the global $(p, \delta, \varepsilon_{\mathrm{ps}})$ of the IQR threshold theorem (Theorem 7.2) in precisely the same way as for GHZ ancillas, mirroring the treatment of ancilla states in conventional fault-tolerance analyses [17, 3, 5].

8.3.5 Ancilla-induced leakage

If GHZ or cluster ancillas are partially realized in the (p, s) sectors (e.g. through parasymmetric / holosymmetric excitations that temporarily leave the Twofold register), they contribute to the leakage parameter $\varepsilon_{\mathrm{ps}}$, in direct analogy with standard treatments of leakage and non-computational levels in fault-tolerant architectures [17, 3, 5]. Let $p^{\mathrm{anc}}_{\mathrm{leak}}$ denote the probability per ancilla that the process enters the (p, s) sector in a way that has support on data PQS (i.e. that the induced channel has nontrivial amplitude outside the register subspace on those sites). If N_{anc} ancillas are used per QEC cycle, then by a simple union bound and the definition of the leakage parameter (Definition 3.15),

$$\Delta\varepsilon_{\mathrm{ps}} \;\lesssim\; N_{\mathrm{anc}}\, p^{\mathrm{anc}}_{\mathrm{leak}}, \tag{592}$$

up to scheme-dependent constants reflecting how ancilla leakage propagates to data PQS. Writing $\varepsilon_{\mathrm{ps,data}}$ for the contribution originating from data-level couplings alone (i.e. leakage in the absence of ancilla usage), we therefore have the schematic decomposition

$$\varepsilon_{\mathrm{ps,total}} \approx \varepsilon_{\mathrm{ps,data}} + \Delta\varepsilon_{\mathrm{ps}}. \tag{593}$$

Justification of the scaling. By Definition 3.15, $\varepsilon_{\mathrm{ps}}$ is defined as a supremum over register input states of the trace distance between the full channel and its restriction to the register subspace. For a single ancilla with leakage probability $p^{\mathrm{anc}}_{\mathrm{leak}}$ (i.e. the probability that the Stinespring isometry has nonzero support in the (p, s)-sector branches that influence data PQS), the induced deviation of the register-restricted channel is bounded by a constant multiple of $p^{\mathrm{anc}}_{\mathrm{leak}}$, since trace distance is upper-bounded by the total probability of leaving the desired subspace. With N_{anc} ancillas per cycle, and under the usual assumption that ancilla-preparation events are independent at the level of this coarse-grained bound, the total deviation adds at most linearly in N_{anc}, giving $\Delta\varepsilon_{\mathrm{ps}} \lesssim N_{\mathrm{anc}} p^{\mathrm{anc}}_{\mathrm{leak}}$, consistent with Lemma 3.19 and standard leakage counting arguments in fault-tolerant schemes [17, 3, 5].

Provided p_{GHZ}, δ_{GHZ}, and $p^{\mathrm{anc}}_{\mathrm{leak}}$ are all sufficiently small (as controlled by the GHZ/cluster robustness polynomials and holosymmetry overlaps in the IQR language; cf. Section 8.3, Definition 5.17, and Theorem 6.12), their contributions to $(p, \delta, \varepsilon_{\mathrm{ps}})$ remain within the thresholds of Theorem 7.2. Hence the use of such resource ancillas inside the code is compatible with the IQR fault-tolerance threshold, as long as their preparation and usage circuits are themselves sufficiently robust in the IQR sense, paralleling the role of ancillas in conventional fault-tolerance analyses [17, 3, 5].

Remark 8.48 (On W-type parasymmetric resources and XY-chain synthesis)**.** In the broader PQS/Fourfold framework there is a natural family of multipartite *parasymmetric* resource states, most prominently the three-site single-excitation W-type configuration

$$|\Psi^{(3)}_{\text{para}}\rangle \;\propto\; |100\rangle + |010\rangle + |001\rangle\,, \tag{594}$$

which is directly related to the canonical three-qubit W state of multipartite entanglement theory [33, 29] and is realizable, for example, as a single-excitation eigenstate of a nearest-neighbour XY chain on three PQS loci, in line with standard XY-chain constructions and single-excitation dynamics [34, 35, 36]. Within the IQR language developed in this article, such W-type parasymmetric resources can in principle be treated using the same structural machinery as the GHZ and cluster ancillas:

- one can associate to them suitable projectors and symmetry operators (noting that W states are not stabilizer states in the strict sense), and from these define robustness functionals and entanglement witnesses in the spirit of Definition 5.17 and Proposition 5.18 and stabilizer-inspired witness constructions [13, 32, 28];
- their dual-history behaviour under local Pauli (and more general, e.g. amplitude-damping) noise can be captured by an extension of the Pauli-channel holosymmetry theorem of Theorem 6.12, yielding holosymmetric valuations of the form $v_{\text{Holo}} = \frac{1}{2}(1+C)$ with C factored into single-site overlaps k_x, just as for GHZ and cluster resources in the Pauli-covariant setting.

However, the present work is organised around a specific QEC architecture in which

1. logical information is encoded in surface-code patches on the emergent Twofold register (cf. Section 5), following the general surface-code paradigm [1, 11, 4, 5], and
2. the only explicit resource ancillas that enter the threshold analysis are GHZ-type and small cluster-state gadgets (cf. the examples in Section 8.3), as in standard syndrome-extraction and lattice-surgery schemes [12, 4, 15].

For this architecture, the IQR fault-tolerance threshold theorem of Theorem 7.2 requires only:

- the existence of ancilla resource states whose robustness functionals $F_S(\Lambda)$ and process overlaps C can be expressed in terms of local noise parameters (e.g. Pauli or depolarising parameters), and
- the ability to absorb those ancilla-induced contributions into the effective $(p, \delta, \varepsilon_{\text{ps}})$ of the global noise model, as described in Lemmas 8.36 and 8.37 and Section 8.3, in line with general resource-theoretic perspectives on fault-tolerant primitives [29, 37].

Both requirements are already satisfied by the GHZ and cluster ancillas treated in detail in Section 8.3, and the proof of Theorem 7.2 does not rely on any specific features of W-type parasymmetric resources or XY-chain dynamics.

For this reason we do *not* develop explicit W-state robustness polynomials or holosymmetry formulae in the present article. These parasymmetric resources form a conceptually rich and technically natural class within the PQS ontology, and their detailed analysis (including XY-chain synthesis, amplitude-damping robustness, and W-specific process holosymmetry) will be treated separately in subsequent work devoted to parasymmetry and resource theory. The general stabilizer and holosymmetry formalism presented here is, however, already sufficient to accommodate such examples without further modification of the IQR threshold framework.

8.4 Summary of Phase 6

The three examples above serve as sanity checks for the abstract IQR threshold framework, showing that it reproduces the expected behaviour of simple codes and ancilla gadgets when specialized to concrete small systems.

- **3-qubit repetition code.** The 3-qubit repetition code realizes the IQR Twofold fibres as physical qubits and takes its bit-flip noise directly from the Twofold block $\Delta\hat{G}_{(0,1)}$ via the effective channel $\Lambda_p^{(X)}$. Under the QEC cycle of Definition 8.5, the logical error probability is
$$p_L^{(\mathrm{rep})} = 3p^2 - 2p^3 = 3p^2 + O(p^3),$$
and the logical Thesis valuation for the correctness proposition φ_{rep} is
$$v_{\mathrm{Th}}\Big(\rho_{\mathrm{out}}; \varphi_{\mathrm{rep}}\Big) = 1 - 3p^2 + O(p^3),$$
exactly matching the textbook scaling for a distance-3 repetition code under bit-flip noise [6, 5]. The dual-history comparison of two repetition gadgets with error rates p_A, p_B yields a holosymmetric valuation
$$v_{\mathrm{Holo}}^{(\mathrm{rep})} = \frac{1}{2}\Big(1 + k^3\Big),$$
with k the single-site Pauli overlap, in direct agreement with the Pauli-channel holosymmetry factorization of Theorem 6.12 specialized to $m_3 = 3$.

- **Tiny 9-qubit surface-code-like patch.** The 3×3 patch with four Z-plaquette stabilizers $S_Z = \langle S_1, S_2, S_3, S_4 \rangle$ provides an explicit instance of the stabilizer weight-distribution machinery underlying Proposition 5.18. The Z-weight distribution of S_Z (Lemma 8.17) leads to the concrete robustness polynomial
$$F_{\mathrm{SC},Z}(\lambda_Z) = \frac{1}{16}\Big(1 + 9\,\lambda_Z^4 + \lambda_Z^5 + 4\,\lambda_Z^6 + \lambda_Z^7\Big),$$
for a Z-only Pauli channel with shrink factor $\lambda_Z = 1 - 2p_Z$ (Definition 8.18 and Proposition 8.19). Under independent local Pauli noise with rate p and distance $d_L = 3$, the logical failure probability on such a minimal planar patch scales as
$$p_L(p) \sim C' p^{(d_L+1)/2} = C' p^2,$$

for some geometry- and decoder-dependent constant C', reproducing the characteristic $O(p^2)$ behaviour of distance-3 surface codes under local noise [1, 4, 5]. This example concretely links the abstract robustness functional $F_{\mathrm{SC},L}$, the stabilizer weight distribution, and the expected distance-controlled scaling of logical errors.

- **GHZ/cluster ancillas and their contribution to $(p, \delta, \varepsilon_{\mathrm{ps}})$.** For GHZ ancillas, the robustness functional

  $$F_{\mathrm{GHZ}}(\Lambda) = \langle \mathrm{GHZ}_n | \Lambda^{\otimes n}(P_{\mathrm{GHZ}}) | \mathrm{GHZ}_n \rangle$$

  reduces, under pure dephasing Λ_μ, to

  $$F_{\mathrm{GHZ}}(\mu) = \frac{1}{2}\Big(1 + \mu^3\Big)$$

  for $n = 3$ (Lemma 8.30), with a sharp entanglement-witness threshold $\mu > 2^{-1/3}$ supplied by Proposition 8.33. For the 4-qubit cluster ancilla $|C_4\rangle$, the stabilizer robustness polynomial under isotropic depolarisation is

  $$F_{C_4}(\eta) = \frac{1}{16}\Big(1 + 2\eta^2 + 8\eta^3 + 5\eta^4\Big)$$

  (Lemma 8.41), while the dual-history overlap for depolarising noise with parameters p_A, p_B in two schedules is

  $$C^{(4)}_{\mathrm{cluster}}(p_A, p_B) = \Big(k_{\mathrm{dep}}(p_A, p_B)\Big)^4, \qquad k_{\mathrm{dep}}(p_A, p_B) = \sqrt{p_A p_B} + \sqrt{(1 - p_A)(1 - p_B)},$$

  giving the holosymmetric valuation $v^{(C_4)}_{\mathrm{Holo}} = \frac{1}{2}(1 + C^{(4)}_{\mathrm{cluster}})$ (Lemma 8.46). Quantities such as $1 - F_{\mathrm{GHZ}}$, $1 - F_{C_4}$ and $(1 - k_{\mathrm{dep}})$ feed directly into the effective data error rate p, the process differential δ, and the leakage parameter $\varepsilon_{\mathrm{ps}}$ via Lemmas 8.36 and 8.37 and the ancilla-leakage discussion of Sections 3 and 8.3, in parallel with ancilla-based error-budget estimates in standard fault-tolerant QEC [17, 3, 12, 5].

Taken together, these concrete cases confirm that the abstract IQR construction recovers the expected behaviour of simple codes and ancilla gadgets:

- logical Thesis valuations reproduce the standard distance-controlled scaling of logical errors (quadratic in p for $d = 3$ toy codes, and exponentially small in d_L in the full surface-code family);
- holosymmetric process valuations reduce to factored Pauli overlaps consistent with the general Pauli-channel holosymmetry theorem;
- resource-ancilla robustness and holosymmetry enter the noise budget precisely through the effective $(p, \delta, \varepsilon_{\mathrm{ps}})$ parameters used in the IQR threshold theorem.

This provides strong internal consistency checks for Phases 1–5 and their synthesis in the full IQR fault-tolerance threshold statement of Theorem 7.2, within the broader IQR/Tetralemma framework of [8].

9 Ambient setting for QEC: triadic diagnostic realized reference configuration and realized weave

In this section we fix the background structures with respect to which all quantum error–correction (QEC) and fault–tolerant quantum computation (FTQC) statements will be made. Throughout, we work inside the IQR framework with a PQS continuum $\mathcal{X}_q$, a Fourfold Dialectic *-algebra $\mathfrak{A}^{(4)}$, and a Holo–State $\omega^{\mathring{\Psi}}$ with GNS triple $(\pi_\Psi, \mathcal{H}_{\mathrm{em}}, |\Omega_\Psi\rangle)$, as developed in the foundational IQR construction of [8] and in line with the standard operator–algebraic GNS framework for quantum field theories and quantum statistical systems [38, 39]. The aim of this section is to isolate a *QEC diagnostic realized reference configuration* which already encodes triadic geometry, a Good–clock, and band–limitation, but does not presuppose any specific dynamical content beyond what is required to talk about QEC primitives (gates, stabilizer measurements, readout and feed–forward), in the sense of modern QEC and FTQC architectures [18, 20, 13, 17, 3, 5, 6].

9.1 QEC diagnostic realized reference configuration

We recall that diagnostic realized reference configurations are the operational configurations on which relational and thermodynamic statements are validated in the IQR/Tetralemma setting [8]. For QEC we need a specialised instance that is adapted to an emergent register, to triadic distances and times, and to a Good–clock, while remaining compatible with standard fault–tolerance requirements [3, 5].

Definition 9.1 (QEC diagnostic realized reference configuration)**.** A *QEC diagnostic realized reference configuration* is a sextuple

$$\mathscr{D}_{\mathrm{QEC}} := \Big(\mathfrak{B},\ \mathfrak{P},\ \mathcal{H}_{\mathrm{phys}},\ \bar{\Sigma}_{\mu\nu},\ C,\ \mathscr{I}_{\mathrm{QEC}}\Big), \tag{595}$$

consisting of the following data:

(a) **PQS domain.** An IQR–regular PQS domain $\mathfrak{B} \subset \mathcal{X}_q$, called the *QEC PQS domain*, such that:

  (a) for every $x \in \mathfrak{B}$ there exists a distinguished PQS $\mathfrak{B}_x \subset \mathfrak{A}^{(4)}$ (the PQS locus at x), as in the general PQS construction of [8];

  (b) $\mathfrak{B}$ lies in the IQR regularity class on $\mathcal{X}_q$ (so that restrictions of the triadic and Fourfold inter–modal correlators, and of the Holo–State, are well–defined) and is large enough to support the emergent qubit registers and ancilla regions used by the QEC protocols under consideration, in analogy with the spatial supports required for stabilizer and surface–code architectures [13, 1, 11, 4, 5].

(b) **Certified plateau.** A nonempty subset $\mathfrak{P} \subset \mathcal{U}_{\mathrm{UV}} \times \mathcal{U}_{\mathrm{IR}}$ of the UV/IR parameter plane, called the *certified plateau*, with the following properties:

  (a) for each $(\lambda_{\mathrm{UV}}, \ell) \in \mathfrak{P}$ there is a band–limiting map $\mathfrak{B}_{\lambda_{\mathrm{UV}},\ell} : \mathcal{B}(\mathcal{H}_{\mathrm{em}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{em}})$, which is completely positive and norm–continuous in $(\lambda_{\mathrm{UV}}, \ell)$, as in the plateau analysis of Chapter 15 of [8];

  (b) along $\mathfrak{P}$ all diagnostic observables relevant for QEC (e.g. stabilizers, local gates, meter effects) admit well–defined plateau limits in the sense of Chapter 15, ensuring controlled UV/IR behaviour of QEC instrumentation in the IQR setting [8].

(c) **Physical Hilbert space.** A closed subspace $\mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{em}}$, called the *physical Hilbert space*, given as the range of a self–adjoint projector $\mathsf{P}_{\text{phys}} : \mathcal{H}_{\text{em}} \to \mathcal{H}_{\text{em}}$, i.e. $\mathcal{H}_{\text{phys}} = \operatorname{Ran} \mathsf{P}_{\text{phys}}$, such that:

  (a) P_{phys} implements the BRST/Nielsen–cleaning appropriate to the theory sector under consideration, in the sense of BRST–type constructions for constrained gauge systems [40, 41, 42];

  (b) all QEC channels, gates, and measurement effects are realised as completely positive maps on $\mathcal{B}(\mathcal{H}_{\text{phys}})$, in line with the standard CPTP/instrument framework for quantum operations [43, 6].

(d) **Realized chronogeometric tensor.** A smooth emergent tensor field $\bar{\Sigma}_{\mu\nu} : \mathfrak{B} \to \mathbb{R}$, the *realized chronogeometric tensor*, of the form

$$\bar{\Sigma}_{\mu\nu}(x) \;=\; \bar{g}_{\mu\nu}(x) + \bar{t}_{\mu\nu}(x), \qquad x \in \mathfrak{B}, \tag{596}$$

where:

  (a) $\bar{g}_{\mu\nu}$ and $\bar{t}_{\mu\nu}$ are obtained as expectation values of the operator–valued metric and temporal tensors $\hat{g}_{\mu\nu}, \hat{t}_{\mu\nu}$ constructed from the triadic distance and time channels, evaluated in the Holo–State and restricted to the plateau $\mathfrak{P}$, in the spirit of relational and operational spacetime reconstructions [44, 45, 8];

  (b) $\bar{\Sigma}_{\mu\nu}$ is nondegenerate on $\mathfrak{B}$ and determines a Levi–Civita connection $\nabla^{(\text{rel})}$ on the relevant tensor bundles over the emergent chronogeometric manifold associated with $\mathfrak{B}$, providing the chronogeometric background relative to which QEC dynamics is timed and localised.

(e) **Good–clock.** A pair $C = (\hat{T}, U_t)$, called a *Good–clock* on $\mathcal{H}_{\text{phys}}$, where:

  (a) $\hat{T}$ is a self–adjoint operator on $\mathcal{H}_{\text{phys}}$, the clock observable;

  (b) $(U_t)_{t\in\mathbb{R}}$ is a strongly continuous one–parameter unitary group on $\mathcal{H}_{\text{phys}}$ with generator $\hat{H}_C$, satisfying the covariance relation $U_t^\dagger\, \hat{T}\, U_t = \hat{T} + t\,\mathbb{I}$ on a dense domain, in line with relational and operational treatments of quantum clocks and time observables [46, 47, 45];

  (c) the integral curves of the chronogeometric vector field defined by $\bar{\Sigma}_{\mu\nu}$ and $\hat{H}_C$ have small temporal holonomy along QEC time scales (Good–clock alignment), so that QEC time intervals can be identified with intervals in the spectrum of $\hat{T}$, as required for time–coherent implementation of discrete QEC cycles and FTQC schedules [6, 5].

(f) **QEC instrumentation.** A collection $\mathscr{I}_{\text{QEC}}$ of completely positive, normal maps on $\mathcal{B}(\mathcal{H}_{\text{phys}})$, called the *QEC instrumentation*, consisting of:

(a) a family $\mathscr{G}_{\mathrm{loc}}$ of *local gate unitaries*, each given by a unitary U on $\mathcal{H}_{\mathrm{phys}}$ with support contained in a finite subset of PQS loci in $\mathfrak{B}$, and acting as a CPTP map $\mathcal{U}(\rho) = U\rho U^{\dagger}$, in the spirit of local gate sets used in stabilizer and surface–code constructions [18, 13, 1, 4, 5];

(b) a family $\mathcal{M}_{\mathrm{stab}}$ of *stabilizer measurement instruments*, each specified by a finite collection $\{\mathcal{M}_{\alpha}\}_{\alpha}$ of completely positive, trace–non–increasing maps on $\mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ such that $\sum_{\alpha} \mathcal{M}_{\alpha}$ is trace–preserving and the associated POVM elements are compatible with the local stabilizer projectors of the emergent code, following the general instrument formalism for quantum measurements [43, 48, 6];

(c) a family $\mathcal{R}_{\mathrm{out}}$ of *readout instruments*, describing final measurements on designated output subsystems of the emergent register and ancillae, e.g. logical readout and syndrome extraction [6, 5];

(d) a class of *feed–forward channels* implementing adaptive operations conditioned on classical records obtained from $\mathcal{M}_{\mathrm{stab}}$ and $\mathcal{R}_{\mathrm{out}}$, realised as CP maps on $\mathcal{B}(\mathcal{H}_{\mathrm{phys}} \otimes \mathcal{H}_{\mathrm{class}})$ with a finite–dimensional classical register $\mathcal{H}_{\mathrm{class}}$, as in standard models of measurement–based feed–forward and fault–tolerant decoding [12, 3, 5].

The following standing convention fixes the ambient setting for all subsequent QEC and threshold statements.

Assumption 9.2 (QEC ambient configuration)**.** *Throughout the remainder of this work, a QEC diagnostic realized reference configuration* $\mathscr{D}_{\mathrm{QEC}}$ *as in Definition 9.1 is fixed once and for all. All notions of* noise channels*,* logical gates*,* stabilizer measurements*,* readout procedures*, and* dual–history (holosymmetric) gadgets *are understood as operations realised by elements of* $\mathscr{I}_{\mathrm{QEC}}$*, acting on* $\mathcal{B}(\mathcal{H}_{\mathrm{phys}})$*, localised in the PQS domain* $\mathfrak{B}$*, and parametrised in time with respect to the Good–clock* C *aligned with the realized chronogeometry* $\bar{\Sigma}_{\mu\nu}$*, within the IQR/Tetralemma framework of [8] and the general CPTP/instrument picture of quantum dynamics and control [43, 6, 5].*

Definition 9.1 fixes, once and for all, the background configuration on which we are allowed to talk about quantum error correction (QEC) and fault–tolerant quantum computation (FTQC) inside the IQR/Tetralemma framework. Concretely, the sextuple $\mathscr{D}_{\mathrm{QEC}} = (\mathfrak{B}, \mathfrak{P}, \mathcal{H}_{\mathrm{phys}}, \bar{\Sigma}_{\mu\nu}, C, \mathscr{I}_{\mathrm{QEC}})$ packages exactly the same structural ingredients that the book uses for PQS domains, certified plateaux, BRST–physical sectors, realized chronogeometry, Good–clocks, and instrumentation, but specialised to QEC primitives.[8, Chs. 1–2, 4, 5, 9, 10, 15] Assumption 9.2 then locks this in as the ambient setting: from that point on, all "noise channels", "logical gates", "stabilizer measurements", "readout procedures", and "dual–history (holosymmetric) gadgets" are, by definition, elements of $\mathscr{I}_{\mathrm{QEC}}$ acting on $\mathcal{B}(\mathcal{H}_{\mathrm{phys}})$, localized in the PQS domain $\mathfrak{B}$, timed by the Good–clock C, and compatible with the realized chronogeometric tensor $\bar{\Sigma}_{\mu\nu}$. This is the direct QEC analogue of the thermodynamic diagnostic realized reference configurations introduced in the relational thermodynamics chapter.[8, Chap. 15]

The PQS domain $\mathfrak{B} \subset \mathcal{X}_q$ is simply a QEC–adapted instance of the PQS arenas used throughout the book.[8, Chs. 1–2] For each $x \in \mathfrak{B}$ there is a distinguished PQS locus $\mathfrak{B}_x \subset \mathfrak{A}^{(4)}$, exactly as in the foundational construction where each locus carries its Fourfold modal fibre and inter–modal correlator. IQR–regularity of $\mathfrak{B}$ just means that the triadic distances, the Fourfold inter–modal correlators, and the Holo–State restrictions behave well enough to support the emergent metric $\hat{g}_{\mu\nu}$, temporal tensor $\hat{t}_{\mu\nu}$, and the associated field and chronogeometric structures, precisely as required in the chronogeometric and matter sector chapters.[8, Chs. 4, 5, 9, 10] The genuinely new requirement for QEC is that $\mathfrak{B}$ be large enough to host the entire QEC patch (data qubits, ancillae, and routing region), in direct analogy with the way the book assumes regions large enough to support graviton– and kymon–bearing sectors or thermodynamic subsystems.[8, Chs. 9–10, 15]

The certified plateau $\mathfrak{P} \subset \mathcal{U}_{\mathrm{UV}} \times \mathcal{U}_{\mathrm{IR}}$ is precisely the same notion of plateau that underpins the emergent chronogeometry and relational thermodynamics in the book.[8, Chaps. 5, 9, 10, 15] For each $(\lambda_{\mathrm{UV}}, \ell) \in \mathfrak{P}$ there is a completely positive, norm–continuous band–limiting map $\mathfrak{B}_{\lambda_{\mathrm{UV}},\ell} : \mathcal{B}(\mathcal{H}_{\mathrm{em}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{em}})$, and along $\mathfrak{P}$ all QEC–relevant observables (stabilizers, local gate unitaries, meter effects) admit well–defined plateau limits. Conceptually this is identical to the use of plateaux in the book to control UV/IR behaviour of triadic distances, $\hat{g}_{\mu\nu}$, $\hat{t}_{\mu\nu}$, retarded kernels, and SynTh meters: the same certified region in the scale plane now guarantees that the QEC instrumentation behaves well under band–limitation, just as thermodynamic and field quantities do.[8, Chs. 5, 9, 10, 15]

The physical Hilbert space $\mathcal{H}_{\mathrm{phys}} = \mathrm{Ran}\, \mathsf{P}_{\mathrm{phys}} \subset \mathcal{H}_{\mathrm{em}}$ is exactly the BRST–physical sector used in the book's treatment of constrained gauge dynamics, chronogeometry, and matter fields.[8, Chs. 9–10] The projector $\mathsf{P}_{\mathrm{phys}}$ implements the BRST/Nielsen cleaning appropriate to the theory sector, so that the physical cohomology $H^0(\hat{Q}_{\mathrm{BRST}})$ is represented on $\mathcal{H}_{\mathrm{phys}}$ and gauge/ghost artefacts are removed. The only additional requirement here is QEC–specific: all QEC channels, gates, and measurement instruments must be realised as completely positive maps on $\mathcal{B}(\mathcal{H}_{\mathrm{phys}})$, which is just the usual FTQC demand that we only manipulate physical degrees of freedom, expressed in the same operator–algebraic/BRST language as the gravitational and matter sectors.[8, Chs. 9–10] Thus $\mathcal{H}_{\mathrm{phys}}$ remains the common stage for graviton, Apeiron, kymon, and now the emergent register used for QEC.

The realized chronogeometric tensor $\bar{\Sigma}_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{t}_{\mu\nu}$ on $\mathfrak{B}$ is not a new object: it is exactly the expectation–level chronogeometry constructed in the book from the triadic distance and time channels via the operator–valued metric $\hat{g}_{\mu\nu}$ and temporal tensor $\hat{t}_{\mu\nu}$, evaluated in the Quantum Holo–State and restricted to the certified plateau.[8, Chs. 4, 5, 9, 10] Nondegeneracy of $\bar{\Sigma}_{\mu\nu}$ and the existence of the Relational Levi–Civita connection $\nabla^{(\mathrm{rel})}$ are precisely the standing assumptions needed throughout the chronogeometric analysis: they underpin the York/TT decompositions, Σ–causal cones, and the propagation of Gaiamiti and Tantrakyma fields.[8, Chs. 9–10] In the QEC context, the only new emphasis is that this same $\bar{\Sigma}_{\mu\nu}$ is now declared to be the chronogeometric background relative to which QEC dynamics is timed and localized, in exactly the same way that it serves as the background for Good–clocked thermodynamics in the relational thermodynamics chapter.[8, Chap. 15]

The Good–clock $C = (\hat{T}, U_t)$ is likewise the same relational clock structure used in the book's discussion of time and thermodynamics.[8, Chap. 15] The clock observable $\hat{T}$ is self–adjoint on $\mathcal{H}_{\mathrm{phys}}$, the one–parameter unitary group U_t has generator $\hat{H}_C$, and together they satisfy the covariance relation $U_t^{\dagger} \hat{T} U_t = \hat{T} + t\,\mathbb{I}$ on a suitable dense domain, in line with the Page–Wootters and relational time constructions.[8, Chs. 4, 15] The Good–clock alignment condition—that the integral curves of the chronogeometric vector field determined by $\bar{\Sigma}_{\mu\nu}$ and $\hat{H}_C$ have small temporal holonomy on the relevant scales—was already required in the book to identify thermodynamic time intervals with intervals in the spectrum of $\hat{T}$.[8, Chap. 15] Here the "times of interest" are discrete QEC cycles and FTQC schedules, but the underlying notion is unchanged: we choose a Good–clock of the same type as in the book and insist that its time parameter aligns cleanly with the step times of the QEC architecture.

Finally, the QEC instrumentation $\mathscr{I}_{\mathrm{QEC}}$ is a QEC–flavoured specialisation of the instrumentation/meters that appear on diagnostic realized reference configurations in the chronogeometric and thermodynamic analysis.[8, Chs. 9–10, 15] It consists of local gate unitaries supported on finite subsets of PQS loci in $\mathfrak{B}$, realised as CPTP maps $\mathcal{U}(\rho) = U\rho U^{\dagger}$; stabilizer measurement instruments $\{\mathcal{M}_{\alpha}\}_{\alpha}$ in the Davies–Ozawa sense, whose POVM elements match the local stabilizers of the emergent code; readout instruments for logical and ancilla outputs; and feed–forward channels implemented as CP maps on $\mathcal{B}(\mathcal{H}_{\mathrm{phys}} \otimes \mathcal{H}_{\mathrm{class}})$ that condition subsequent operations on classical records. All of this uses the same CPTP/instrument formalism that the book employs for quantum measurements, SynTh meters, and coarse–grained dynamics, now applied to QEC primitives rather than thermodynamic or field–theoretic observables.[8, Chs. 9–10, 15]

Assumption 9.2 then plays the role of an "alignment lock": one such QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$ is fixed once and for all, and every subsequent use of terms like noise channel, logical gate, stabilizer measurement, readout procedure, or dual–history gadget is understood to refer to operations realised by elements of $\mathscr{I}_{\mathrm{QEC}}$ on $\mathcal{B}(\mathcal{H}_{\mathrm{phys}})$, localised in $\mathfrak{B}$, timed by the Good–clock C, and embedded in the IQR/Tetralemma framework established in the book.[8, Chs. 1–2, 4, 5, 9, 10, 15] In this sense, the article's "QEC diagnostic realized reference configuration" is a direct specialisation of the diagnostic realized reference configurations used for chronogeometry and thermodynamics: it is tailored to QEC and FTQC, but all upstream structures—PQS continuum and loci, Holo–State, certified plateaux, BRST–physical sector, realized chronogeometry, and Good–clock—are carried over unchanged.

9.2 Triadic and Fourfold inter–modal correlators in the QEC domain

We briefly recall how the triadic and Fourfold inter–modal correlators of the full IQR framework restrict to the QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$ of Definition 9.1, and how they generate the realized chronogeometry $\bar{\Sigma}_{\mu\nu}$ and the emergent Twofold register on the PQS domain $\mathfrak{B}$, in alignment with the constructions in Chapters 1,2,4,5,9,10 and 15 of [8].

9.2.1 Triadic distance and time channels on $\mathfrak{B}$

Definition 9.3 (Triadic inter–modal channels)**.** On the PQS continuum $\mathcal{X}_q$, the IQR framework associates to the Holo–State $\omega^{\mathring{\Psi}}$ a family of triadic inter–modal channels

$$\hat{\mathcal{D}},\ \hat{\mathcal{T}} : C_0^{\infty}(\mathcal{X}_q^3) \longrightarrow \mathcal{B}(\mathcal{H}_{\mathrm{em}}), \tag{597}$$

the *triadic distance* and *triadic time* channels, constructed from the underlying Fourfold Dialectic *-algebra $\mathfrak{A}^{(4)}$ as in Chapters 1,2 and 4 of [8]. For a test function $f \in C_0^{\infty}(\mathcal{X}_q^3)$ we write

$$\hat{\mathcal{D}}[f] = \int_{\mathcal{X}_q^3} f(x,y,z)\,\hat{\mathcal{D}}(x,y,z)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z), \tag{598}$$

$$\hat{\mathcal{T}}[f] = \int_{\mathcal{X}_q^3} f(x,y,z)\,\hat{\mathcal{T}}(x,y,z)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z), \tag{599}$$

where $\hat{\mathcal{D}}(x,y,z)$ and $\hat{\mathcal{T}}(x,y,z)$ are operator–valued kernels with four modal indices $a,b,c \in \{0,1,p,s\}$ suppressed for brevity, and μ is the canonical IQR volume measure on $\mathcal{X}_q$ [8]. The action of these channels on $\mathcal{H}_{\mathrm{em}}$ encodes, at the operator level, the relational distance and time information between triples of PQS loci.

Definition 9.4 (Restriction to the QEC domain and plateau)**.** Given a QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$ with PQS domain $\mathfrak{B}$ and certified plateau $\mathfrak{P}$ (Definition 9.1), the triadic channels are *restricted* to $\mathfrak{B} \times \mathfrak{B} \times \mathfrak{B}$ and to UV/IR parameters $(\lambda_{\mathrm{UV}}, \ell) \in \mathfrak{P}$ via the band–limiting maps $\mathfrak{B}_{\lambda_{\mathrm{UV}},\ell}$ of Definition 9.1(b). This yields the plateau–regularised kernels

$$\hat{\mathcal{D}}_{\mathfrak{B},\mathfrak{P}}(x,y,z) := \mathfrak{B}_{\lambda_{\mathrm{UV}},\ell}\Big(\hat{\mathcal{D}}(x,y,z)\Big)\Big|_{x,y,z\in\mathfrak{B}}, \tag{600}$$

$$\hat{\mathcal{T}}_{\mathfrak{B},\mathfrak{P}}(x,y,z) := \mathfrak{B}_{\lambda_{\mathrm{UV}},\ell}\Big(\hat{\mathcal{T}}(x,y,z)\Big)\Big|_{x,y,z\in\mathfrak{B}}, \tag{601}$$

which are well–defined and norm–continuous in $(\lambda_{\mathrm{UV}}, \ell) \in \mathfrak{P}$, in the sense of the plateau analysis of Chapter 15 in [8].

Definition 9.5 (Emergent metric and temporal tensors from triadic channels)**.** For each $x \in \mathfrak{B}$, choose local coordinates (x^μ) adapted to the realized chronogeometry induced by the Holo–State (Chapters 5 and 9 of [8]). The operator–valued metric and temporal tensors $\hat{g}_{\mu\nu}(x)$, $\hat{t}_{\mu\nu}(x)$ entering Definition 9.1(d) are obtained as coincident–limit Hessians of the plateau–regularised triadic kernels:

$$\hat{g}_{\mu\nu}(x) := -\frac{1}{2}\,\frac{\partial^2}{\partial y^\mu \partial z^\nu}\hat{\mathcal{D}}_{\mathfrak{B},\mathfrak{P}}(x,y,z)\Big|_{y=z=x}, \tag{602}$$

$$\hat{t}_{\mu\nu}(x) := -\frac{1}{2}\,\frac{\partial^2}{\partial y^\mu \partial z^\nu}\hat{\mathcal{T}}_{\mathfrak{B},\mathfrak{P}}(x,y,z)\Big|_{y=z=x}, \tag{603}$$

where the derivatives are taken in the sense of operator–valued distributions on $\mathcal{H}_{\mathrm{em}}$ and evaluated in the Holo–State $\omega^{\mathring{\Psi}}$. Their Holo–State expectations define the realized chronogeometric tensor $\bar{\Sigma}_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{t}_{\mu\nu}$ of Definition 9.1(d):

$$\bar{g}_{\mu\nu}(x) := \omega^{\mathring{\Psi}}\Big(\hat{g}_{\mu\nu}(x)\Big), \qquad \bar{t}_{\mu\nu}(x) := \omega^{\mathring{\Psi}}\Big(\hat{t}_{\mu\nu}(x)\Big). \tag{604}$$

Remark 9.6. The above coincident–limit construction is the QEC–adapted specialization of the general triadic chronogeometric reconstruction in the full IQR framework, where relational distance and time are extracted from triadic correlations of PQS observables in the Holo–State [8]. The restriction to $\mathfrak{B}$ and $\mathfrak{P}$ ensures that all QEC time and distance scales lie in a regime where the emergent chronogeometry is sufficiently smooth and stable to support discrete QEC cycles and FTQC schedules, as required in Definition 9.1.

Definition 9.7 (Synge world function)**.** Let (M, g) be a time–orientable Lorentzian manifold and let $\mathcal{N} \subset M \times M$ be a normal neighbourhood of the diagonal $\{(x, x) \mid x \in M\}$ such that any pair $(x, x') \in \mathcal{N}$ is joined by a unique geodesic. The *Synge world function* $\sigma : M \times M \supset \mathcal{N} \to \mathbb{R}$ is defined by

$$\sigma(x, x') := \tfrac{1}{2}\, s(x, x')^2, \tag{605}$$

where $s(x, x')$ is the signed geodesic distance between x and x'. In a convex normal neighbourhood one has the standard coincidence–limit relation

$$g_{\mu\nu}(x) \;=\; -\,\nabla_\mu \nabla_{\nu'} \sigma(x, x')\big|_{x'=x}\,, \tag{606}$$

which expresses the spacetime metric in terms of the Hessian of σ at coincidence.[2]

In the IQR/Tetralemma setting we generalise this picture in two directions. First, geometric information is encoded in operator–valued distributions on a Hilbert space $\mathcal{H}_{\mathrm{em}}$, in the spirit of algebraic quantum field theory.[38, 39] Second, the basic relational data are triadic and inter–modal, rather than merely two–point, and are encoded in the distance and time channels $\hat{\mathcal{D}}, \hat{\mathcal{T}}$ on the PQS continuum $\mathcal{X}_q$, as recalled in Definition 9.3 and in the foundational IQR construction of [8]. The phrase "operator–valued Synge world function" is a convenient shorthand for this operator–algebraic, triadic generalisation of Synge's world function.

Definition 9.8 (Operator–valued Synge world kernels in the IQR setting)**.** Let $\mathcal{X}_q$ be the PQS continuum, let $\omega^{\mathring{\Psi}}$ be a Holo–State with GNS triple $(\pi_\Psi, \mathcal{H}_{\mathrm{em}}, |\Omega_\Psi\rangle)$, and let $\mathscr{D}_{\mathrm{QEC}}$ be a QEC diagnostic realized reference configuration with PQS domain $\mathfrak{B}$ and certified plateau $\mathfrak{P}$ (Definition 9.1). A pair of operator–valued distributions

$$\hat{\mathcal{D}}_{\mathfrak{B},\mathfrak{P}},\; \hat{\mathcal{T}}_{\mathfrak{B},\mathfrak{P}} : C_0^\infty(\mathfrak{B}^3) \longrightarrow \mathcal{B}(\mathcal{H}_{\mathrm{em}}) \tag{607}$$

obtained from the triadic distance and time channels by restriction to $\mathfrak{B}$ and plateau regularisation (Definition 9.4) will be called *operator–valued Synge world kernels* for distance and time if they satisfy the following properties.

(i) For each Holo–State $\omega^{\mathring{\Psi}}$ the expectation values

$$D_\Psi(x, y, z) := \omega^{\mathring{\Psi}}\Big(\hat{\mathcal{D}}_{\mathfrak{B},\mathfrak{P}}(x, y, z)\Big), \qquad T_\Psi(x, y, z) := \omega^{\mathring{\Psi}}\Big(\hat{\mathcal{T}}_{\mathfrak{B},\mathfrak{P}}(x, y, z)\Big) \tag{608}$$

are smooth scalar kernels in a neighbourhood of the diagonal $\{(x, y, z) \in \mathfrak{B}^3 \mid x = y = z\}$, and are real–valued and symmetric under interchange of the "argument" loci in the sense appropriate to the IQR relational geometry.[8]

(ii) The operator–valued metric and temporal tensors $\hat{g}_{\mu\nu}(x)$, $\hat{t}_{\mu\nu}(x)$ defined by the coincidence–limit Hessians

$$\hat{g}_{\mu\nu}(x) := -\frac{1}{2}\,\frac{\partial^2}{\partial y^\mu \partial z^\nu}\hat{\mathcal{D}}_{\mathfrak{B},\mathfrak{P}}(x, y, z)\bigg|_{y=z=x}\,, \tag{609}$$

$$\hat{t}_{\mu\nu}(x) := -\frac{1}{2}\,\frac{\partial^2}{\partial y^\mu \partial z^\nu}\hat{\mathcal{T}}_{\mathfrak{B},\mathfrak{P}}(x, y, z)\bigg|_{y=z=x}\,, \tag{610}$$

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[2] See, for example, Synge's original treatment [49] and the modern expositions in [50, 51].

exist as operator–valued distributions on $\mathcal{H}_{\rm em}$ and have Holo–State expectations

$$\bar g_{\mu\nu}(x) := \omega^{\mathring{\Psi}}\Big(\hat g_{\mu\nu}(x)\Big), \qquad \bar t_{\mu\nu}(x) := \omega^{\mathring{\Psi}}\Big(\hat t_{\mu\nu}(x)\Big) \tag{611}$$

which coincide with the realized metric and temporal tensors of the IQR chronogeometric reconstruction, so that $\bar\Sigma_{\mu\nu} = \bar g_{\mu\nu} + \bar t_{\mu\nu}$ is the realized chronogeometric tensor of Definition 9.1(d).

Remark 9.9 (Interpretation)**.** Classically, Synge's world function $\sigma(x, x')$ is a scalar kernel on $M \times M$ whose Hessian at coincidence reproduces the metric and thereby encodes the local geometry of spacetime.[49, 51] In the IQR/Tetralemma framework, the kernels $\hat{\mathcal{D}}_{\mathfrak{B},\mathfrak{P}}(x, y, z)$ and $\hat{\mathcal{T}}_{\mathfrak{B},\mathfrak{P}}(x, y, z)$ play an analogous role, but at the level of operator–valued distributions on the emergent Hilbert space $\mathcal{H}_{\rm em}$.[8] Their Holo–State expectations D_Ψ and T_Ψ are the scalar triadic distance and time kernels, and their coincidence–limit Hessians yield the operator–valued metric and temporal tensors $\hat g_{\mu\nu}, \hat t_{\mu\nu}$, whose expectations in turn define the realized chronogeometric tensor $\bar\Sigma_{\mu\nu}$. The phrase "operator–valued Synge world functions" is therefore to be understood as referring to these operator–valued world–function kernels: they generalise Synge's classical construction to the operator–algebraic, triadic, inter–modal setting of IQR, while still serving the same structural purpose of generating the metric (and its temporal companion) via coincidence–limit Hessians.

Definition 9.3 is fully consistent with the way the book builds relational geometry from triadic data. In the book, the composite triadic operator–valued distance $D^2(x, x')$ is assembled from three "axes" $d_{\rm corr}, d_{\rm vac}, d_{\rm spec}$, each of which is itself constructed from Fourfold correlators, transporters, and spectral data.[8, Sec. 5.2] The operator–valued metric $\hat g_{\mu\nu}(x)$ is then defined from the second variation of D^2, and the triadic decomposition $\hat g_{\mu\nu} = \hat g^{\rm corr}_{\mu\nu} + \hat g^{\rm vac}_{\mu\nu} + \hat g^{\rm spec}_{\mu\nu}$ is extracted as in [8, Ch. 5] In this subsection, rather than starting from D^2 directly, one introduces triadic inter–modal channels $\hat{\mathcal{D}}, \hat{\mathcal{T}} : C_0^\infty(\mathcal{X}_q^3) \to \mathcal{B}(\mathcal{H}_{\rm em})$ with kernels $\hat{\mathcal{D}}(x, y, z)$ and $\hat{\mathcal{T}}(x, y, z)$ carrying modal indices $a, b, c \in \{0, 1, p, s\}$. These kernels function as operator–valued Synge world functions for triadic distance and time, built from the same Fourfold Dialectic *-algebra $\mathfrak{A}^{(4)}$ and Holo–State $\omega^{\mathring{\Psi}}$ that Chapters 1, 2, and 4 use to define PQS fibres, correlators, and transporters.[8, Chs. 1–2, 4] The triadic structure that the book usually packages into D^2 and its components $\hat d_a$ is thus made explicit as a pair of channels $\hat{\mathcal{D}}, \hat{\mathcal{T}}$, which is a refinement rather than a departure. A minor stylistic refinement (not a mathematical change) would be to describe these as "operator–valued kernels with modal indices $a, b, c \in \{0, 1, p, s\}$, one at each PQS locus, each index taking four possible modal values", which avoids any ambiguity between "four indices" and "indices with four values".

Definition 9.4 then applies the same plateau machinery that the book develops for thermodynamics and chronogeometry.[8, Ch. 15] There, band–limiting maps $\mathfrak{B}_{\lambda_{\rm UV},\ell}$ on $\mathcal{B}(\mathcal{H}_{\rm em})$, norm–continuous in $(\lambda_{\rm UV}, \ell)$, are used to define a certified plateau $\mathfrak{P} \subset \mathcal{U}_{\rm UV} \times \mathcal{U}_{\rm IR}$, and various emergent objects (SynTh meters, thermodynamic observables, chronogeometric tensors) are only trusted in that plateau regime after band–limiting.[8, Chs. 5, 9, 10, 15] The present subsection does exactly the same for the triadic distance and time channels: restricting to $x, y, z \in \mathfrak{B}$ and applying $\mathfrak{B}_{\lambda_{\rm UV},\ell}$ yields plateau–regularised kernels $\hat{\mathcal{D}}_{\mathfrak{B},\mathfrak{P}}(x, y, z)$

and $\hat{\mathcal{T}}_{\mathfrak{B},\mathfrak{P}}(x,y,z)$, which are norm–continuous in $(\lambda_{\mathrm{UV}},\ell)\in\mathfrak{P}$. This is precisely the QEC–specialised use of the certified plateau introduced in Definition 9.1(b) and lines up with the role $\mathfrak{P}$ plays in the chronogeometric and thermodynamic chapters: it ensures that all distance and time data used for QEC are drawn from the same plateau–regularised regime that underpins $\bar{\Sigma}_{\mu\nu}$ and the thermodynamic diagnostic realized reference configuration.[8, Chs. 5, 9, 10, 15]

Definition 9.5 makes explicit how these triadic channels generate the operator–valued metric and temporal tensors $\hat{g}_{\mu\nu},\hat{t}_{\mu\nu}$. The formulas

$$\hat{g}_{\mu\nu}(x) := -\frac{1}{2}\,\frac{\partial^2}{\partial y^\mu \partial z^\nu}\hat{\mathcal{D}}_{\mathfrak{B},\mathfrak{P}}(x,y,z)\bigg|_{y=z=x}, \qquad \hat{t}_{\mu\nu}(x) := -\frac{1}{2}\,\frac{\partial^2}{\partial y^\mu \partial z^\nu}\hat{\mathcal{T}}_{\mathfrak{B},\mathfrak{P}}(x,y,z)\bigg|_{y=z=x}$$

are the triadic analogue of deriving the metric from the second variation of the Synge world function. In Chapter 5, the metric is defined as the second variation of the composite triadic distance $D^2(x,x')$, built from operator–valued distances $\hat{d}_a(x,x')$.[8, Sec. 5.2] Formally, this is the operator–level counterpart of the familiar relation $g_{\mu\nu}\sim -\frac{1}{2}\partial_\mu\partial'_\nu\sigma(x,x')|_{x'=x}$. Here, $\hat{\mathcal{D}}_{\mathfrak{B},\mathfrak{P}}(x,y,z)$ plays the role of the triadic world–function kernel from which $\hat{g}_{\mu\nu}$ is extracted by a coincident–limit Hessian, and the minus sign together with the factor $\frac{1}{2}$ are exactly in line with the "second variation of D^2" phrasing in the book.[8, Ch. 5] Evaluating in the Holo–State,

$$\bar{g}_{\mu\nu}(x) := \omega^{\mathring{\boldsymbol{\Psi}}}\Big(\hat{g}_{\mu\nu}(x)\Big), \qquad \bar{t}_{\mu\nu}(x) := \omega^{\mathring{\boldsymbol{\Psi}}}\Big(\hat{t}_{\mu\nu}(x)\Big),$$

recovers the expectation–level relational manifold $(\mathcal{X}_q,\bar{g}_{\mu\nu},\bar{t}_{\mu\nu})$ introduced in the book.[8, Ch. 5] Thus, in the QEC context, this subsection simply spells out explicitly the construction that Chapter 5 describes more invariantly in terms of triadic distances and their second variations. The triadic decomposition into correlational, vacuum, and spectral contributions remains implicit: $\hat{\mathcal{D}}$ and $\hat{\mathcal{T}}$ are built from the axes $d_{\mathrm{corr}},d_{\mathrm{vac}},d_{\mathrm{spec}}$ and their temporal counterparts, and the identity $\hat{g}_{\mu\nu}=\hat{g}^{\mathrm{corr}}_{\mu\nu}+\hat{g}^{\mathrm{vac}}_{\mu\nu}+\hat{g}^{\mathrm{spec}}_{\mu\nu}$ continues to hold in this restricted setting [8].

The closing remark correctly identifies this coincident–limit construction as the QEC–adapted specialization of the general triadic chronogeometric reconstruction. It mirrors the way the full theory extracts relational distances and times from triadic correlations in the Holo–State, first to define the operator–valued metric and temporal tensors and then to obtain the realized chronogeometric tensor $\bar{\Sigma}_{\mu\nu}=\bar{g}_{\mu\nu}+\bar{t}_{\mu\nu}$.[8, Chs. 4–5] The additional restriction to the QEC PQS domain $\mathfrak{B}$ and the certified plateau $\mathfrak{P}$ ensures that all QEC time and distance scales lie in the regime where Chapter 15's "certified plateau" guarantees smooth and stable chronogeometry, now specifically calibrated to support discrete QEC cycles and FTQC schedules.[8, Ch. 15] This dovetails cleanly with Definition 9.1(d): there $\bar{\Sigma}_{\mu\nu}$ was taken as inherited from the book's chronogeometric construction, and here it is shown explicitly that, after restriction to $\mathfrak{B}$ and $\mathfrak{P}$, the triadic channels produce exactly that same $\bar{\Sigma}_{\mu\nu}$. Nothing new is being assumed; the subsection simply makes explicit how the QEC ambient configuration inherits its chronogeometric data from the global IQR construction.

9.2.2 Fourfold correlators and the emergent Twofold register

Definition 9.10 (Fourfold inter–modal correlator in the QEC domain)**.** The full IQR framework associates to the Fourfold Dialectic *-algebra $\mathfrak{A}^{(4)}$ and Holo–State $\omega^{\mathring{\boldsymbol{\Psi}}}$ a Fourfold inter–modal correlator

$$\hat{G}_\Lambda : \mathcal{X}_q^4 \longrightarrow \mathcal{B}(\mathcal{H}_{\mathrm{em}}), \tag{612}$$

with modal indices $a_1, a_2, a_3, a_4 \in \{0, 1, p, s\}$ and dependence on a collective scale parameter Λ (Chapter 10 of [8]). In the QEC diagnostic realized reference configuration we restrict $\hat{G}_\Lambda$ to the PQS domain $\mathfrak{B}$ and plateau $\mathfrak{P}$, obtaining

$$\hat{G}_{\Lambda;\mathfrak{B},\mathfrak{P}}(x_1, x_2, x_3, x_4) := \mathfrak{B}_{\lambda_{\mathrm{UV}},\ell}\Big(\hat{G}_\Lambda(x_1, x_2, x_3, x_4)\Big)\Big|_{x_i\in\mathfrak{B}}, \tag{613}$$

for $(\lambda_{\mathrm{UV}}, \ell) \in \mathfrak{P}$. The corresponding ideal correlator $\hat{G}^0_\Lambda$ and noise kernel $\Delta\hat{G} := \hat{G}_\Lambda - \hat{G}^0_\Lambda$ are defined as in Definition 4.1, now understood as restricted to $\mathfrak{B}$ and $\mathfrak{P}$.

Definition 9.11 (Modal block decomposition and Twofold sector)**.** At each PQS locus $x \in \mathfrak{B}$ the modal fibre $\mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}}$ carries a distinguished orthonormal Fourfold basis $\{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$, and decomposes as

$$\mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}} = \mathcal{H}_{(0,1),x} \oplus \mathcal{H}_{(p,s),x}, \tag{614}$$

with $\mathcal{H}_{(0,1),x} := \mathrm{span}\{|0\rangle_x, |1\rangle_x\}$ the emergent Twofold sector, as constructed in Section 3 and Lemma 3.7. In this basis, the restricted Fourfold correlator admits a block decomposition

$$\hat{G}_{\Lambda;\mathfrak{B},\mathfrak{P}} = \begin{pmatrix} \hat{G}^{(0,1)}_{\Lambda;\mathfrak{B},\mathfrak{P}} & \hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}} \\ \hat{G}^{\mathrm{mix}\,\dagger}_{\Lambda;\mathfrak{B},\mathfrak{P}} & \hat{G}^{(p,s)}_{\Lambda;\mathfrak{B},\mathfrak{P}} \end{pmatrix}, \tag{615}$$

where $\hat{G}^{(0,1)}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ is the Twofold block, $\hat{G}^{(p,s)}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ the parasymmetric block, and $\hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ contains off–block couplings between Twofold and parasymmetric sectors.

Fourfold inter–modal correlator in the QEC domain. Definition 9.10 is the QEC/plateau–restricted version of the Fourfold inter–modal correlator $\hat{G}_\Lambda$ that the book introduces when it constructs the Apeiron/kymon sectors and the Fourfold Dialectic *-algebra.[8, Ch. 10] In the book, this correlator is defined as a map

$$\hat{G}_\Lambda : \mathcal{X}_q^{\times n} \longrightarrow \mathcal{B}(\mathcal{H}_{\mathrm{em}}),$$

equipped with modal indices $a_i \in \{0, 1, p, s\}$ that encode Fourfold correlations between PQS loci and carry an explicit dependence on a collective scale parameter Λ (e.g. a renormalisation or correlation scale).[8, Chs. 4, 9, 10] Writing the domain as $\mathcal{X}_q^4$ in the present QEC setting simply reflects the fact that we are focusing on a four–point kernel (the expectation of a product of four PQS selectors or transporters) relevant for the QEC noise analysis; this does not conflict with the two–point shorthand $\hat{G}_\Lambda(\mathfrak{B}_x, \mathfrak{B}_{x'})$ used earlier, which is naturally viewed as a contraction or special case of the same underlying Fourfold object. The subsequent restriction to the PQS domain $\mathfrak{B}$ and certified plateau $\mathfrak{P}$ is exactly parallel to the treatment of triadic channels and to the way the book handles SynTh meters, chronogeometric tensors, and thermodynamic quantities: one applies the band–limiting map $\mathfrak{B}_{\lambda_{\mathrm{UV}},\ell}$ (as in Definition 9.1(b)) and then restricts all loci to lie in $\mathfrak{B}$, obtaining

$$\hat{G}_{\Lambda;\mathfrak{B},\mathfrak{P}}(x_1,x_2,x_3,x_4) := \mathfrak{B}_{\lambda_{\mathrm{UV}},\ell}\Big(\hat{G}_\Lambda(x_1,x_2,x_3,x_4)\Big)\Big|_{x_i\in\mathfrak{B}},$$

which is norm–continuous in $(\lambda_{\mathrm{UV}},\ell)\in\mathfrak{P}$ in precisely the sense used in the plateau analysis of Chapter 15.[8, Ch. 15] The ideal correlator $\hat{G}^0_\Lambda$ and the noise kernel $\Delta\hat{G} := \hat{G}_\Lambda - \hat{G}^0_\Lambda$ are then defined exactly as in Definition 4.1 of the article, now explicitly understood as restricted to $\mathfrak{B}$ and $\mathfrak{P}$. This tightens the scope without changing the object: $\hat{G}_\Lambda$ is the same Fourfold correlator constructed in the book, and the same plateau machinery that regulates $\hat{\mathcal{D}}$ and $\hat{\mathcal{T}}$ is being applied to it.[8, Chs. 5, 9, 10, 15]

Modal block decomposition and the emergent Twofold sector. Definition 9.11 is a QEC–adapted restatement of the modal fibre structure that the book develops and that the article has already exploited in its earlier phases. At each PQS locus $\mathfrak{B}_x$ one has a modal fibre $\mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}}$ carrying a distinguished Fourfold basis $\{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$, arising from the Fourfold Dialectic *-algebra $\mathfrak{A}^{(4)}$ and the GNS construction on the Holo–State.[8, Chs. 1–2, 4] Earlier lemmas in the article show that this basis can be chosen orthonormal and that the fibre decomposes as

$$\mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}} = \mathcal{H}_{(0,1),x} \oplus \mathcal{H}_{(p,s),x}, \qquad \mathcal{H}_{(0,1),x} := \mathrm{span}\{|0\rangle_x, |1\rangle_x\}, \qquad \mathcal{H}_{(p,s),x} := \mathrm{span}\{|p\rangle_x, |s\rangle_x\},$$

with $\mathcal{H}_{(0,1),x}$ identified as the emergent Twofold/qubit sector and $\mathcal{H}_{(p,s),x}$ as the parasymmetric/holo sector, in line with the book's Fourfold-to-Twofold reduction and the emergent register construction.[8, Chs. 4, 9, 10] Given this splitting, it is natural to view the restricted Fourfold correlator $\hat{G}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ as a 2×2 block operator in modal index space with respect to the decomposition "Twofold vs. (p,s)", which is exactly what Definition 9.11 does:

$$\hat{G}_{\Lambda;\mathfrak{B},\mathfrak{P}} = \begin{pmatrix} \hat{G}^{(0,1)}_{\Lambda;\mathfrak{B},\mathfrak{P}} & \hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}} \\ \hat{G}^{\mathrm{mix}\,\dagger}_{\Lambda;\mathfrak{B},\mathfrak{P}} & \hat{G}^{(p,s)}_{\Lambda;\mathfrak{B},\mathfrak{P}} \end{pmatrix}.$$

Here, $\hat{G}^{(0,1)}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ is the block with all legs in the Twofold subspace $\mathcal{H}_{(0,1)}$, and, after the further reductions carried out in subsequent sections, it is this block that controls the effective local noise on the emergent qubit register. The block $\hat{G}^{(p,s)}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ is the purely parasymmetric/holo sector, largely invisible at the logical level but important for leakage and for the triadic provenance analysis that appears later. The mixed block $\hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ contains the off–block couplings between $\mathcal{H}_{(0,1)}$ and $\mathcal{H}_{(p,s)}$; these couplings are precisely what feed into the leakage parameter $\varepsilon_{\mathrm{ps}}$ and its triadic refinements $\varepsilon^{(a)}_{\mathrm{ps}}$ in the Triadic–Fourfold Noise Class defined downstream. In this way, the block decomposition implements at the level of the Fourfold correlator the same Twofold/(p,s) separation that underlies the emergent register, the leakage–control lemmas in Phase 1, and the triadic leakage bounds in the later fault–tolerance analysis, all of which are grounded in the Fourfold modal structure developed in the book.[8, Chs. 4, 9, 10]

Lemma 9.12 (Emergent register and leakage from $\Delta\hat{G}$)**.** *In the QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$, the restriction of the ideal and actual Fourfold correlators to the Twofold block on $\mathfrak{B}$ determines:*

(a) the emergent Twofold register $\mathcal{H}^{(L)}_{\mathrm{reg}} = \bigotimes_{x_j \in \mathcal{P}_L} \mathcal{H}_{(0,1),x_j}$ *on any PQS patch* $\mathcal{P}_L \subset \mathfrak{B}$*, as in Section 3 and Definition 5.2; and*

(b) the effective local noise kernel $\Delta\hat{G}_{(0,1)}$ *on the Twofold sector, whose smallness and clustering properties (Assumption 4.5) control the deviation between the actual and ideal register channels and determine:*

- *the effective local Pauli noise on* $\mathcal{H}^{(L)}_{\mathrm{reg}}$*, summarized in Corollary 4.20 and Equation (362);*
- *the leakage parameter* $\varepsilon_{\mathrm{ps}}$*, via the off–block components* $\hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ *and the block–Kraus structure of Assumption 3.17, as in Lemmas 3.18 and 3.19.*

Proof idea. The existence of the emergent Twofold fibres $\mathcal{H}_{(0,1),x} \subset \mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}}$ and their smooth bundle structure over $\mathfrak{B}$ is established in Section 3 by analysing the coincident–limit behaviour of the Fourfold correlator in the modal basis and identifying a distinguished qubit–like subspace at each PQS locus. Taking tensor products over a finite patch $\mathcal{P}_L \subset \mathfrak{B}$ yields $\mathcal{H}^{(L)}_{\mathrm{reg}}$, in direct analogy with the construction of emergent registers in [8].

The noise kernel $\Delta\hat{G} := \hat{G}_\Lambda - \hat{G}^0_\Lambda$ restricted to the Twofold block defines, via its Choi representation, the deviation between the actual and ideal local channels at each Twofold site; smallness and clustering of $\Delta\hat{G}_{(0,1)}$ (Assumption 4.5) imply the diamond–norm bounds on local and global register channels derived in Lemmas 4.10, 4.11, 4.15 and 4.18, leading to Corollary 4.20. The off–block components $\hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ similarly control the norm of the off–block Kraus operators in the full channel on $\mathcal{H}_{\mathrm{em}}$; the block–Kraus assumption (Assumption 3.17) and Lemma 3.18 then bound the leakage parameter $\varepsilon_{\mathrm{ps}}$, with Lemma 3.19 showing its stability under composition of QEC cycles. Together, these arguments establish the claims.

Proof. We split the argument into two parts, corresponding to (a) and (b).

(a) Emergent Twofold register. By Definition 9.11, at each PQS locus $x \in \mathfrak{B}$ the modal fibre $\mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}}$ carries an orthonormal Fourfold basis $\{|0\rangle_x, |1\rangle_x, |p\rangle_x, |s\rangle_x\}$ and decomposes as

$$\mathcal{H}^{(\mathfrak{B}_x)}_{\mathrm{mod}} = \mathcal{H}_{(0,1),x} \oplus \mathcal{H}_{(p,s),x}, \qquad \mathcal{H}_{(0,1),x} := \mathrm{span}\{|0\rangle_x, |1\rangle_x\}, \quad \mathcal{H}_{(p,s),x} := \mathrm{span}\{|p\rangle_x, |s\rangle_x\}. \tag{616}$$

In Section 3, the existence and uniqueness (up to local phases) of the Twofold subspaces $\mathcal{H}_{(0,1),x}$ are obtained by analysing the coincident–limit structure of the Fourfold correlator $\hat{G}_\Lambda$ in the modal basis: the spectral projectors associated with the distinguished qubit–like eigenstructure of the restricted correlator select $|0\rangle_x$ and $|1\rangle_x$ at each locus, and their smooth dependence on x over $\mathfrak{B}$ yields a smooth rank–two subbundle $\mathcal{H}_{(0,1)} \subset \mathcal{H}_{\mathrm{mod}}$ (see Lemma 3.7 and Section 3). This construction only uses the ideal (noise–free) part of the Fourfold correlator, since the identification of $\mathcal{H}_{(0,1),x}$ is made at the level of the ideal emergent dynamics (cf. the emergent register constructions in the book[8, Chs. 4, 9, 10]).

For any finite PQS patch $\mathcal{P}_L \subset \mathfrak{B}$, the emergent register Hilbert space is then defined by

$$\mathcal{H}^{(L)}_{\mathrm{reg}} := \bigotimes_{x_j \in \mathcal{P}_L} \mathcal{H}_{(0,1),x_j}, \tag{617}$$

as in Definition 5.2. Since each factor $\mathcal{H}_{(0,1),x_j}$ is determined by the ideal Twofold block of $\hat{G}_\Lambda$ at x_j, the entire tensor product $\mathcal{H}^{(L)}_{\mathrm{reg}}$ is determined by the restriction of the ideal correlator to the Twofold block on the patch $\mathcal{P}_L$. This establishes item (a).

(b) Effective local noise and leakage from $\Delta\hat{G}_{(0,1)}$. We now fix a finite patch $\mathcal{P}_L \subset \mathfrak{B}$ and consider a single QEC cycle implemented by instrumentation $\mathscr{I}_{\mathrm{QEC}}$ in the ambient configuration $\mathscr{D}_{\mathrm{QEC}}$ (see Assumption 9.2). For each such cycle, the actual and ideal dynamics on the emergent register are described by completely positive trace–preserving (CPTP) channels $\mathcal{E}^{(L)}$ and $(\mathcal{E}^{(L)})^0$ on $\mathcal{B}(\mathcal{H}^{(L)}_{\mathrm{reg}})$, obtained by restricting the full dynamics on $\mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ to the Twofold sector and tracing out the parasymmetric degrees of freedom. By the general Choi–Jamiołkowski correspondence for quantum channels (see, e.g., [6] and Lemma 4.10), each channel $\mathcal{E}^{(L)}$ is in bijection with a Choi operator $J(\mathcal{E}^{(L)})$ on $\mathcal{H}^{(L)}_{\mathrm{reg}} \otimes \mathcal{H}^{(L)}_{\mathrm{reg}}$, and similarly for $(\mathcal{E}^{(L)})^0$.

By construction of the IQR noise class, the difference between the actual and ideal four–point correlators can be written as $\Delta\hat{G} := \hat{G}_\Lambda - \hat{G}^0_\Lambda$, and when we restrict to the Twofold block we obtain $\Delta\hat{G}_{(0,1)} := \hat{G}^{(0,1)}_{\Lambda;\mathfrak{B},\mathfrak{P}} - (\hat{G}^0)^{(0,1)}_{\Lambda;\mathfrak{B},\mathfrak{P}}$. The Choi–correlator relation of Lemma 4.10 then implies that the difference of the Choi operators,

$$\Delta J^{(L)} := J(\mathcal{E}^{(L)}) - J((\mathcal{E}^{(L)})^0), \tag{618}$$

is obtained by integrating $\Delta\hat{G}_{(0,1)}$ against fixed test functions that encode the QEC instrumentation, the Good–clock, and the geometry of the patch $\mathcal{P}_L$. Linear dependence of $\Delta J^{(L)}$ on $\Delta\hat{G}_{(0,1)}$ together with the smallness and clustering bounds of the IQR noise class (see Assumption 4.5) therefore yield an operator–norm bound of the form

$$\left\|\Delta J^{(L)}\right\| \le C_L\,\epsilon_0, \tag{619}$$

where ϵ_0 is the fundamental smallness parameter controlling $\Delta\hat{G}_{(0,1)}$ and C_L depends polynomially on the size of the patch (through the number of interaction terms allowed by the clustering lengths and times). Applying the diamond–vs–Choi estimates of Lemma 4.11 then yields

$$\left\|\mathcal{E}^{(L)} - (\mathcal{E}^{(L)})^0\right\|_\diamond \le C'_L\,\epsilon_0, \tag{620}$$

for some constant C'_L depending only on the patch geometry and the instrumentation. This establishes that $\Delta\hat{G}_{(0,1)}$ controls the deviation between the actual and ideal register channels in diamond norm.

Assumption 4.5 furthermore requires that $\Delta\hat{G}_{(0,1)}$ be short–range correlated in space and time and sufficiently weak that, at the single–site level, it lies in the local IQR regime considered in Lemma 4.15. At the level of one or a few neighbouring sites, Lemma 4.15 then provides a local Pauli approximation to the associated channels, with error bounded in terms of ϵ_0. The approximate tensor–product structure ensured by the clustering assumptions and formalised in Lemma 4.18 allows one to extend this local approximation to the full patch $\mathcal{P}_L$, yielding an effective local Pauli noise model on $\mathcal{H}^{(L)}_{\mathrm{reg}}$. This is the model later summarized in Corollary 4.20 and Equation (362); crucially, its parameters are fixed by the smallness and clustering properties of $\Delta\hat{G}_{(0,1)}$ through the above chain of Choi and diamond–norm bounds.

We now turn to leakage. By the modal block decomposition of Definition 9.11, the full Fourfold correlator on the QEC domain splits into a Twofold block, a parasymmetric block, and a mixed block $\hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}}$. The corresponding full channel on $\mathcal{H}_{\mathrm{em}}$ can be represented in Kraus form, and Assumption 3.17 asserts that this Kraus representation can be chosen block–structured with respect to the decomposition $\mathcal{H}_{\mathrm{phys}} = \mathcal{H}_{(0,1)} \oplus \mathcal{H}_{(p,s)}$, so that the off–block Kraus operators are linearly controlled by the mixed block of the correlator. More precisely, the Choi–correlator map of Lemma 4.10, applied now to the full channel, shows that the off–block part of the Choi operator is determined by integrating $\hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ against the same test functions. Under the block–Kraus hypothesis, the norm of these off–block Choi components bounds the sum of the squared norms of the off–block Kraus operators. Lemma 3.18 makes this precise and identifies a leakage parameter $\varepsilon_{\mathrm{ps}}$ such that

$$\varepsilon_{\mathrm{ps}} \simeq \sum_k \left\| K_k^{\mathrm{off}} \right\|^2 \lesssim C'' \sup_{x_i \in \mathcal{P}_L} \left\| \hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}}(x_1, x_2, x_3, x_4) \right\|, \tag{621}$$

where K_k^{off} denotes the off–block Kraus operators and C'' is a constant depending on the instrumentation and patch geometry.

Finally, Lemma 3.19 shows that when the IQR clustering assumptions hold and the QEC cycles are sufficiently local, the leakage parameter $\varepsilon_{\mathrm{ps}}$ is stable under composition of QEC cycles, in the sense that it grows at most linearly (or sublinearly) with the number of cycles on the relevant time scales. Taken together, these facts establish that $\hat{G}^{\mathrm{mix}}_{\Lambda;\mathfrak{B},\mathfrak{P}}$ determines the leakage parameter $\varepsilon_{\mathrm{ps}}$, and that $\Delta\hat{G}_{(0,1)}$ determines the effective local noise on $\mathcal{H}^{(L)}_{\mathrm{reg}}$ via its influence on the Choi and diamond norms of the register channels. This completes the proof of item (b) and of the lemma. □

Remark 9.13. Definitions 9.3–9.11 and Lemma 9.12 summarize, in QEC–adapted form, how the full triadic and Fourfold inter–modal correlator structure of the IQR framework feeds into:

- the realized chronogeometry $\bar{\Sigma}_{\mu\nu}$ and Good–clock alignment used to time and localise QEC schedules; and
- the emergent Twofold register, effective local Pauli noise, and leakage parameter $\varepsilon_{\mathrm{ps}}$ which enter the IQR fault–tolerance threshold theorem (Theorem 7.2).

Thus the QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$ is not an additional structure imposed by hand, but rather a controlled restriction of the full triadic/Fourfold correlator machinery of IQR to a regime where QEC primitives are well–defined and fault–tolerance can be analysed within the IQR/Tetralemma framework.

9.3 Realized weave configuration as the carrier

Within a fixed QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$ (Definition 9.1), we now specify the realized weave configuration which serves as the carrier for all QEC noise modelling, Fourfold inter–modal correlators, and emergent register constructions, in the IQR/PQS sense of [8] and in keeping with the operator–algebraic GNS framework [38, 39].

Definition 9.14 (QEC–realized weave configuration)**.** Let $\mathscr{D}_{\mathrm{QEC}}$ be a QEC diagnostic realized reference configuration with physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$ and realized chronogeometric tensor $\bar{\Sigma}_{\mu\nu}$. A *QEC–realized weave configuration* on $\mathscr{D}_{\mathrm{QEC}}$ is a tuple

$$\mathcal{I}^{[\mathscr{S}]} := \Big(\mathcal{A},\, r;\, P_x(\lambda_{\mathrm{UV}}),\, K_{\mathrm{IR}}^{(r)};\, \mathsf{P}_{\mathrm{phys}};\, \bar{\Sigma}_{\mu\nu}\Big), \tag{622}$$

with the following components and properties:

(a) **Admissible settings.** $\mathcal{A}$ is a (nonempty) set of *admissible holosymmetric settings*, encoding choices of internal gauges, reference frames, and instrument conventions compatible with the algebra $\mathfrak{A}^{(4)}$ and the Holo–State, in the sense of the holosymmetry framework of [8]. Elements of $\mathcal{A}$ act as *–automorphisms of $\mathfrak{A}^{(4)}$ that preserve the physical subspace $\mathcal{H}_{\mathrm{phys}}$, as is standard for admissible dynamical/kinematical symmetries in algebraic QFT [39, 38].

(b) **Infrared tuning.** r is an infrared tuning parameter taking values in a label set $\mathcal{L}_{\mathrm{IR}}$. For each $r \in \mathcal{L}_{\mathrm{IR}}$ there exists a completely positive, normal map

$$K_{\mathrm{IR}}^{(r)} :\ \mathcal{B}(\mathcal{H}_{\mathrm{phys}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}}), \tag{623}$$

called an *IR window*, such that:

  (a) $K_{\mathrm{IR}}^{(r)}$ is idempotent and contractive in the operator norm;

  (b) the family $\{K_{\mathrm{IR}}^{(r)}\}_{r\in\mathcal{L}_{\mathrm{IR}}}$ is compatible with the certified plateau $\mathfrak{P}$ in the sense that plateau limits of QEC observables exist after application of $K_{\mathrm{IR}}^{(r)}$, as in the UV/IR band–limiting constructions of [8].

(c) **UV band projectors.** For each PQS locus $x \in \mathfrak{B}$ and each UV parameter λ_{UV} admitted by the plateau $\mathfrak{P}$, there is a projector

$$P_x(\lambda_{\mathrm{UV}}) :\ \mathcal{H}_{\mathrm{phys}} \to \mathcal{H}_{\mathrm{phys}}, \tag{624}$$

called the *UV band projector at* x, with the properties:

  (a) $P_x(\lambda_{\mathrm{UV}})$ is self–adjoint and idempotent;

  (b) for fixed x, the map $\lambda_{\mathrm{UV}} \mapsto P_x(\lambda_{\mathrm{UV}})$ is strongly continuous and monotone in the sense of spectral inclusion;

  (c) the band–limiting map $\mathfrak{B}_{\lambda_{\mathrm{UV}},r}$ associated with $(\lambda_{\mathrm{UV}}, r) \in \mathfrak{P}$ can be written, up to physically irrelevant phases, in terms of the family $\{P_x(\lambda_{\mathrm{UV}})\}_{x\in\mathfrak{B}}$ and $K_{\mathrm{IR}}^{(r)}$, as in the plateau–projection constructions of [8].

(d) **Physical projector.** $\mathsf{P}_{\mathrm{phys}} : \mathcal{H}_{\mathrm{em}} \to \mathcal{H}_{\mathrm{em}}$ is the BRST/Nielsen–cleaning projector appearing in Definition 9.1, with $\mathcal{H}_{\mathrm{phys}} = \operatorname{Ran} \mathsf{P}_{\mathrm{phys}}$. It is required that

$$\mathsf{P}_{\mathrm{phys}}\, P_x(\lambda_{\mathrm{UV}}) = P_x(\lambda_{\mathrm{UV}})\, \mathsf{P}_{\mathrm{phys}}, \qquad \mathsf{P}_{\mathrm{phys}}\, K_{\mathrm{IR}}^{(r)} = K_{\mathrm{IR}}^{(r)}\, \mathsf{P}_{\mathrm{phys}}, \tag{625}$$

for all $x \in \mathfrak{B}$ and all $(\lambda_{\mathrm{UV}}, r) \in \mathfrak{P}$. This ensures compatibility of the BRST/Nielsen cleaning [40, 41, 42] with the UV/IR band–limiting structures used to define the plateau and QEC instrumentation [8].

(e) **Triad projectors.** There exist three commuting orthogonal projectors

$$\Pi^{(\mathrm{spec})},\ \Pi^{(\mathrm{vac})},\ \Pi^{(\mathrm{corr})}\ \in\ \mathcal{B}(\mathcal{H}_{\mathrm{phys}}), \tag{626}$$

called the *spectral*, *vacuum*, and *correlational* triad projectors, such that:

(a) they are mutually orthogonal and resolve the identity:

$$\Pi^{(a)}\Pi^{(b)} = \delta_{ab}\Pi^{(a)}, \qquad \sum_{a\in\{\mathrm{spec,vac,corr}\}} \Pi^{(a)} = \mathbf{1}_{\mathcal{H}_{\mathrm{phys}}}; \tag{627}$$

(b) they commute with the physical projector and the band–limiting structures:

$$[\Pi^{(a)}, \mathsf{P}_{\mathrm{phys}}] = 0, \quad [\Pi^{(a)}, P_x(\lambda_{\mathrm{UV}})] = 0, \quad [\Pi^{(a)}, K_{\mathrm{IR}}^{(r)}] = 0, \tag{628}$$

for all $a \in \{\mathrm{spec}, \mathrm{vac}, \mathrm{corr}\}$, $x \in \mathfrak{B}$, and $(\lambda_{\mathrm{UV}}, r) \in \mathfrak{P}$;

(c) they provide a triadic provenance decomposition of all QEC observables and Fourfold inter–modal correlators by orthogonal compression, e.g. $X = \sum_a \Pi^{(a)} X \Pi^{(a)}$ for any QEC–relevant operator X on $\mathcal{H}_{\mathrm{phys}}$, as in the triadic provenance and Tetralemma semantics of [8].

(f) **Compatibility with chronogeometry and Good–clock.** Let $C = (\hat{T}, U_t)$ be the Good–clock of Definition 9.1. The realized chronogeometric tensor $\bar{\Sigma}_{\mu\nu}$ and the Good–clock are said to be compatible with $\mathcal{I}^{[\mathscr{S}]}$ if:

(a) the Levi–Civita connection $\nabla^{(\mathrm{rel})}$ determined by $\bar{\Sigma}_{\mu\nu}$ has vanishing covariant derivative on the triad projectors in the sense that any induced spacetime dependence of $\Pi^{(a)}$ is covariantly constant along the PQS domain $\mathfrak{B}$;

(b) the Good–clock evolution preserves the triad decomposition, i.e.

$$U_t^{\dagger}\,\Pi^{(a)}\,U_t = \Pi^{(a)}, \qquad \forall\, t \in \mathbb{R}, \quad a \in \{\mathrm{spec}, \mathrm{vac}, \mathrm{corr}\}; \tag{629}$$

(c) the QEC cycle durations defined in terms of the spectrum of $\hat{T}$ satisfy the triadic QEC time–window inequalities (to be specified later), ensuring that correlation, vacuum, and spectral dynamics are well separated on QEC time scales, in the sense of relational/clock–based dynamics [46, 44, 45, 47] and compatible with discrete QEC cycle structure [6, 5].

Remark 9.15 (Carrier role and QC–only stance)**.** The QEC–realized weave configuration $\mathcal{I}^{[\mathscr{S}]}$ serves as the *carrier* for:

- the full Fourfold inter–modal correlator $\hat{G}_{\Lambda}$ restricted to the physical, band–limited space $\mathcal{H}_{\mathrm{phys}}$, as developed in the IQR/Fourfold formalism of [8];
- the emergent Twofold register constructed from PQS loci in $\mathfrak{B}$, which will host surface–code and stabilizer architectures [13, 1, 11, 4, 5];

- and all noise channels, logical operations, and dual–history holosymmetric gadgets used in the threshold analysis, in the style of holosymmetric process comparisons introduced in [8].

In the present, quantum–computing–focused development we *do not* introduce or make use of additional projectors or structures that resolve gravitational spin–2 (graviton) or temporal spin–1 (Chrono–Kymon) sectors. Those objects may exist at the level of the full theory, but all QEC and fault–tolerance results in this work depend only on:

(a) the band/BRST–cleaned physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$,

(b) the triad projectors $\Pi^{(\mathrm{spec})}, \Pi^{(\mathrm{vac})}, \Pi^{(\mathrm{corr})}$,

(c) and the realized chronogeometry $\bar{\Sigma}_{\mu\nu}$ together with the Good–clock C,

which suffice for the IQR threshold analysis of surface–code–based FTQC [1, 4, 5, 8].

Assumption 9.16 (Fixed QEC–realized weave configuration)**.** *In addition to Assumption 9.2, we assume that a QEC–realized weave configuration $\mathcal{I}^{[\mathscr{S}]}$ as in Definition 9.14 has been fixed once and for all. All Fourfold inter–modal correlators, emergent register constructions, and effective noise channels considered in the sequel are defined as objects on $\mathcal{H}_{\mathrm{phys}}$ relative to this $\mathcal{I}^{[\mathscr{S}]}$, within the IQR/Tetralemma framework of [8] and the standard CPTP/instrument description of quantum operations and noise [43, 6, 5].*

10 Triadic–Fourfold noise class

In the correlation–only setting of Assumption 4.5, the IQR noise model was expressed in terms of a single correlation distance d_{corr}, a correlation time t_{corr}, and the Twofold block of the deviation kernel $\Delta\hat{G}$ of the inter–modal correlator, within the general IQR framework of [8]. In the present work we extend this to a triadic and Fourfold resolution adapted to the QEC–realized weave configuration $\mathcal{I}^{[\mathscr{S}]}$ (Definition 9.14), so that all noise parameters are defined on the band/BRST–cleaned physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$ with triadic provenance tags.

10.1 Full fourfold correlator and triad resolution

We begin by recalling the Fourfold decomposition of the modal fibre at each PQS locus and introducing the associated block structure for the inter–modal correlator on the physical Hilbert space, following the modal/fibre construction of the IQR/PQS framework [8] and the fixed QEC ambient configuration of Assumption 9.2.

Definition 10.1 (Modal fibre decomposition and Twofold/(p,s) blocks)**.** Let Assumptions 9.2 and 9.16 hold. For each PQS locus $\mathfrak{B}_x$, $x \in \mathfrak{B}$, the modal fibre $\mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)}$ admits an orthogonal decomposition

$$\mathcal{H}_{\mathrm{mod}}^{(\mathfrak{B}_x)} = \mathcal{H}_{(0,1),x} \oplus \mathcal{H}_{(p,s),x}, \tag{630}$$

where:

1. $\mathcal{H}_{(0,1),x}$ is the Twofold (Thesis/AntiThesis) subspace, spanned by orthonormal states $|0\rangle_x, |1\rangle_x$, as constructed via the coincident–limit inner product in [8];
2. $\mathcal{H}_{(p,s),x}$ is the complementary parasymmetric/holo subspace, spanned by orthonormal states $|p\rangle_x, |s\rangle_x$.

For each pair of loci $x, x' \in \mathfrak{B}$, we write $\hat{G}_\Lambda^{\alpha\beta}(x, x')$ for the Fourfold inter–modal correlator components with indices $\alpha, \beta \in \{0, 1, p, s\}$, acting on $\mathcal{H}_{\text{phys}}$ in the GNS representation [38, 39, 8]. Grouping the indices according to the decomposition above yields a 2×2 block operator

$$\hat{G}_\Lambda(x, x') \ := \ \begin{pmatrix} \hat{G}_{(0,1)\to(0,1)}(x, x') & \hat{G}_{(0,1)\to(p,s)}(x, x') \\ \hat{G}_{(p,s)\to(0,1)}(x, x') & \hat{G}_{(p,s)\to(p,s)}(x, x') \end{pmatrix}, \tag{631}$$

where, for instance, $\hat{G}_{(0,1)\to(0,1)}(x, x')$ denotes the component of the correlator that takes the Twofold sector at x' into the Twofold sector at x, and $\hat{G}_{(0,1)\to(p,s)}(x, x')$ denotes the component from the Twofold sector at x' into the (p, s) sector at x, and similarly for the remaining blocks.

The QEC–realized weave configuration $\mathcal{I}^{[\mathscr{S}]}$ (Definition 9.14) provides, in addition, a triad of projectors on $\mathcal{H}_{\text{phys}}$, which we now use to refine the block structure, in line with the triadic provenance decomposition of the Tetralemma semantics [8].

Definition 10.2 (Triad–resolved correlator blocks)**.** Let $\Pi^{(\text{spec})}, \Pi^{(\text{vac})}, \Pi^{(\text{corr})}$ be the spectral, vacuum, and correlational triad projectors on $\mathcal{H}_{\text{phys}}$ associated with the QEC–realized weave configuration $\mathcal{I}^{[\mathscr{S}]}$ (Definition 9.14). For any of the block components $\hat{G}_{\bullet\to\bullet}(x, x')$ of the full Fourfold correlator as in Definition 10.1, and for each triad label $a \in \{\text{spec}, \text{vac}, \text{corr}\}$, we define the *triad–resolved block* by

$$\hat{G}^{(a)}_{\bullet\to\bullet}(x, x') \ := \ \Pi^{(a)}\, \hat{G}_{\bullet\to\bullet}(x, x')\, \Pi^{(a)}, \qquad a \in \{\text{spec}, \text{vac}, \text{corr}\}. \tag{632}$$

In particular, for each pair of loci $x, x' \in \mathfrak{B}$ we obtain a triad decomposition of the full block matrix,

$$\hat{G}_\Lambda(x, x') \ = \sum_{a\in\{\text{spec},\text{vac},\text{corr}\}} \hat{G}^{(a)}_\Lambda(x, x'), \tag{633}$$

where

$$\hat{G}^{(a)}_\Lambda(x, x') \ := \ \begin{pmatrix} \hat{G}^{(a)}_{(0,1)\to(0,1)}(x, x') & \hat{G}^{(a)}_{(0,1)\to(p,s)}(x, x') \\ \hat{G}^{(a)}_{(p,s)\to(0,1)}(x, x') & \hat{G}^{(a)}_{(p,s)\to(p,s)}(x, x') \end{pmatrix}. \tag{634}$$

Definition 10.3 (Ideal correlator and noise kernel)**.** An *ideal Fourfold inter–modal correlator* $\hat{G}^0_\Lambda(x, x')$ on $\mathcal{I}^{[\mathscr{S}]}$ is a correlator which implements the intended (noiseless) dynamics of a single QEC cycle on the emergent register and its ancillae, subject to the same band–limiting and BRST conditions as $\hat{G}_\Lambda(x, x')$, in the sense of the IQR noise model [8]. The *noise kernel* is the operator–valued kernel

$$\Delta\hat{G}(x, x') \ := \ \hat{G}_\Lambda(x, x') - \hat{G}^0_\Lambda(x, x'), \tag{635}$$

with the same block structure as in Definition 10.1. Its triad–resolved components are defined analogously by

$$\Delta\hat{G}^{(a)}_{\bullet\to\bullet}(x,x') \;:=\; \Pi^{(a)}\,\Delta\hat{G}_{\bullet\to\bullet}(x,x')\,\Pi^{(a)}, \qquad a\in\{\mathrm{spec},\mathrm{vac},\mathrm{corr}\}, \tag{636}$$

so that for each block $\Delta\hat{G}_{\bullet\to\bullet}(x,x')$ we have the decomposition

$$\Delta\hat{G}_{\bullet\to\bullet}(x,x') \;=\; \sum_{a\in\{\mathrm{spec},\mathrm{vac},\mathrm{corr}\}} \Delta\hat{G}^{(a)}_{\bullet\to\bullet}(x,x'). \tag{637}$$

Remark 10.4 (Triadic provenance tags)**.** Definitions 10.2 and 10.3 provide, for each sector of the Fourfold inter–modal correlator, a triadic provenance decomposition: spectral, vacuum, and correlational contributions are obtained by orthogonal compression with the projectors $\Pi^{(\mathrm{spec})},\Pi^{(\mathrm{vac})},\Pi^{(\mathrm{corr})}$, as in the general triadic provenance bookkeeping of [8]. In particular, every block of the noise kernel $\Delta\hat{G}$ carries three provenance tags (spec/vac/corr), which will be used in the sequel to define the Triadic–Fourfold Noise Class governing the effective QEC noise model, and to relate it to standard clustered/correlated noise treatments in fault–tolerant quantum error correction [52, 2, 3, 5].

10.2 Triadic–fourfold noise class

We now collect the structures introduced in Sections 10.1 and 10.3 into a single definition of the noise models admissible for our QEC analysis. This *sharpens* the earlier correlation–only IQR noise class of Assumption 4.5: the latter is recovered as a coarse–grained limit in which triadic provenance and the full Fourfold block structure are ignored, while the present Triadic–Fourfold Noise Class keeps track of

- the Twofold versus (p,s) Fourfold blocks,
- the spectral/vacuum/correlational triad provenance tags, and
- the dependence on the realized chronogeometry $\bar{\Sigma}_{\mu\nu}$ and Good–clock $C=(\hat{T},U_t)$.

This refinement remains in the spirit of local, weakly correlated noise models used in rigorous threshold theorems for FTQC [2, 3, 5].

Definition 10.5 (Triadic–Fourfold Noise Class)**.** Let $\mathscr{D}_{\mathrm{QEC}}$ be a QEC diagnostic realized reference configuration and $\mathcal{I}^{[\mathscr{S}]}$ a QEC–realized weave configuration as in Definitions 9.1 and 9.14. Consider a single QEC cycle implemented on a finite PQS configuration $\mathcal{P}\subset\mathfrak{B}$, and let $\hat{G}_\Lambda(x,x')$ be the corresponding fourfold inter–modal correlator on $\mathcal{H}_{\mathrm{phys}}$, with ideal counterpart $\hat{G}^0_\Lambda(x,x')$ and noise kernel $\Delta\hat{G}(x,x')$ as in Definition 10.3.

We say that the noise model for this QEC cycle belongs to the *Triadic–Fourfold Noise Class*, and we write $\Delta\hat{G}\in\mathcal{N}_{\text{tri-4F}}$, if the following conditions hold:

(a) **Realized weave origin.** The kernel $\Delta\hat{G}(x,x')$ is defined on the QEC PQS domain $\mathfrak{B}$ and arises from the difference of two fourfold inter–modal correlators $\hat{G}_\Lambda,\hat{G}^0_\Lambda$ which are both realised on the same QEC–realized weave configuration $\mathcal{I}^{[\mathscr{S}]}$. In particular, the band–limiting maps $\mathfrak{B}_{\lambda_{\mathrm{UV}},\ell}$, the physical projector $\mathsf{P}_{\mathrm{phys}}$, and the triad projectors $\Pi^{(\mathrm{spec})},\Pi^{(\mathrm{vac})},\Pi^{(\mathrm{corr})}$ act consistently on both $\hat{G}_\Lambda$ and $\hat{G}^0_\Lambda$, as in the general IQR/PQS construction of [8].

(b) **Triadic clustering.** The triad–resolved kernels $\Delta\hat{G}^{(a)}_{\bullet\to\bullet}(x,x')$ satisfy the spatial and temporal clustering bounds of Assumption 10.9 for a triadic spatial distance system $\{d_a\}$ and triadic times $t_{\mathrm{corr}}, t_{\mathrm{vac}}, t_{\mathrm{spec}}$ as in Definition 10.8. That is, for each $a \in \{\mathrm{corr}, \mathrm{vac}, \mathrm{spec}\}$ there exist positive constants ξ_a, C_a, C'_a such that

$$\left\|\Delta\hat{G}^{(a)}_{\bullet\to\bullet}(x,x')\right\| \;\le\; C_a \,\exp\!\Big(-\frac{d_a(x,x')}{\xi_a}\Big), \tag{638}$$

for all $x, x' \in \mathfrak{B}$, and

$$\left\|\Delta\hat{G}^{(a)}_{\bullet\to\bullet}(x,t;\,x',t')\right\| \;\le\; C'_a \,\exp\!\Big(-\frac{|t-t'|}{t_a}\Big), \tag{639}$$

for all temporal separations $|t-t'| \ge t_a$, where $t_a \in \{t_{\mathrm{corr}}, t_{\mathrm{vac}}, t_{\mathrm{spec}}\}$ is the corresponding triadic time. This mirrors exponential clustering and Lieb–Robinson–type locality bounds in local quantum many–body systems [53, 54, 55], now refined by triadic provenance.

(c) **Triadic QEC time window.** The QEC cycle duration Δt_{QEC} associated with the Good–clock $C = (\hat{T}, U_t)$ satisfies the triadic QEC time–window inequality of Definition 10.10, namely

$$t_{\mathrm{corr}} \;\ll\; \Delta t_{\mathrm{QEC}} \;\ll\; t_{\mathrm{vac}},\; t_{\mathrm{spec}}. \tag{640}$$

In particular, correlational fluctuations are fast compared to the QEC cycle, whereas vacuum and spectral structures are quasi–static on the time scale of many cycles, as in standard Markovian approximations for local noise in threshold analyses [2, 3, 5, 9].

(d) **Small Twofold–preserving block.** The Twofold–preserving block of the noise kernel is small in operator norm on the QEC patch. More precisely, there exists a constant $\varepsilon_{(0,1)} > 0$ such that

$$\sup_{x,x'\in\mathcal{P}}\left\|\Delta\hat{G}_{(0,1)\to(0,1)}(x,x')\right\| \;\le\; \varepsilon_{(0,1)}. \tag{641}$$

Equivalently, each triad–resolved Twofold block obeys

$$\sup_{x,x'\in\mathcal{P}}\left\|\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}(x,x')\right\| \;\le\; \varepsilon^{(a)}_{(0,1)}, \qquad a \in \{\mathrm{corr}, \mathrm{vac}, \mathrm{spec}\}, \tag{642}$$

for suitable nonnegative constants $\varepsilon^{(a)}_{(0,1)}$ with $\varepsilon_{(0,1)} \le \sum_a \varepsilon^{(a)}_{(0,1)}$. This is the fourfold/triadic refinement of the "small local channel perturbation" condition underlying effective Pauli noise models in the IQR framework and in conventional FTQC [2, 5].

(e) **Small cross blocks and triad–controlled leakage.** The cross blocks of the noise kernel between the Twofold and parasymmetric/holo sectors are small in operator norm on the QEC patch and admit a triadic decomposition which will govern leakage. Concretely, there exist nonnegative constants $\varepsilon^{(a)}_{\mathrm{ps}}$, $a \in \{\mathrm{corr}, \mathrm{vac}, \mathrm{spec}\}$, such that

$$\sup_{x,x'\in\mathcal{P}}\left\|\Delta\hat{G}^{(a)}_{(0,1)\to(p,s)}(x,x')\right\| + \sup_{x,x'\in\mathcal{P}}\left\|\Delta\hat{G}^{(a)}_{(p,s)\to(0,1)}(x,x')\right\| \;\le\; \varepsilon^{(a)}_{\mathrm{ps}}, \tag{643}$$

and the total cross–block size obeys

$$\sup_{x,x'\in\mathcal{P}}\Big(\Big\|\Delta\hat{G}_{(0,1)\to(p,s)}(x,x')\Big\|+\Big\|\Delta\hat{G}_{(p,s)\to(0,1)}(x,x')\Big\|\Big)\ \le\ \varepsilon_{\mathrm{ps}},\tag{644}$$

with $\varepsilon_{\mathrm{ps}}\le\sum_a\varepsilon_{\mathrm{ps}}^{(a)}$. The constants $\varepsilon_{\mathrm{ps}}^{(a)}$ will later be seen to control distinct contributions to leakage out of, and back into, the emergent Twofold register from the correlational, vacuum, and spectral channels respectively, in analogy with leakage models in FTQC [17, 3, 5].

(f) **Plateau compatibility and stationarity on QEC time scales.** For each triad label a and each block of $\Delta\hat{G}^{(a)}$, band–limited plateau limits exist after application of the UV/IR maps $\mathfrak{B}_{\lambda_{\mathrm{UV}},\ell}$ and $K_{\mathrm{IR}}^{(\ell)}$ associated with $\mathcal{I}^{[\mathscr{S}]}$ and $\mathscr{D}_{\mathrm{QEC}}$, and are stationary over QEC cycle times in the sense that the triad–resolved kernels depend only on time differences $(t-t')$ within each cycle. This ensures that the effective noise per QEC cycle is well–defined and time–translation invariant with respect to the Good–clock C.

(g) **Approximate Pauli covariance on the correlational Twofold sector.** On the correlational triad sector $a=\mathrm{corr}$, the Twofold–to–Twofold noise kernel $\Delta\hat{G}^{(\mathrm{corr})}_{(0,1)\to(0,1)}$ satisfies the approximate Pauli–covariance condition of Assumption 4.13 with small error $\epsilon_{\mathrm{twirl}}$. In particular, there exist local single–site Pauli channels Λ_j on the emergent Twofold fibres such that the associated local channels $\mathcal{N}_j^{(1)}$ constructed from $\Delta\hat{G}^{(\mathrm{corr})}$ obey

$$\Big\|\mathcal{N}_j^{(1)}-\Lambda_j\Big\|_\diamond\le\epsilon_{\mathrm{loc}},\qquad\epsilon_{\mathrm{loc}}=O(\epsilon_{0,\mathrm{corr}}+\epsilon_{\mathrm{twirl}}),\tag{645}$$

as in Lemma 4.15 and Corollary 4.20. Here $\epsilon_{0,\mathrm{corr}}$ is a correlational smallness parameter consistent with the bounds above.

The set of all noise kernels $\Delta\hat{G}$ satisfying conditions *(a)–(h)* above will be denoted by $\mathcal{N}_{\text{tri-4F}}$ and referred to as the *Triadic–Fourfold Noise Class.*

Lemma 10.6 (Coarse–graining to the correlation–only IQR noise class)**.** *Let* $\Delta\hat{G}\in\mathcal{N}_{\text{tri-4F}}$ *be a Triadic–Fourfold noise kernel in the sense of Definition 10.5. Then there exist finite constants*

$$\xi>0,\qquad t_{\mathrm{corr}}>0,\qquad\epsilon_0>0,$$

and a spatial distance d_{corr} *on* $\mathfrak{B}$ *(for instance the relational distance induced by* $\bar{\Sigma}_{\mu\nu}$ *on the QEC patch* $\mathcal{P}$*) such that the total Twofold–to–Twofold noise kernel satisfies the clustered bound*

$$\Big\|\Delta\hat{G}_{(0,1)\to(0,1)}(x,t;\,x',t')\Big\|\ \le\ \epsilon_0\ \exp\!\Big(-\frac{d_{\mathrm{corr}}(x,x')}{\xi}-\frac{|t-t'|}{t_{\mathrm{corr}}}\Big),\tag{646}$$

for all $x,x'\in\mathfrak{B}$ *and all* t,t' *within a single QEC cycle. In particular, the correlation–only IQR noise assumption* (Assumption 4.5) *holds for the same QEC cycle with parameters* $(\xi,t_{\mathrm{corr}},\epsilon_0)$.

Proof. By Definition 10.3 and Definition 10.5, the full Twofold block of the noise kernel decomposes into its triad–resolved pieces as

$$\Delta\hat{G}_{(0,1)\to(0,1)}(x,t;\,x',t')=\sum_{a\in\{\mathrm{spec},\mathrm{vac},\mathrm{corr}\}}\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}(x,t;\,x',t').$$

Taking operator norms and using the triangle inequality gives

$$\left\|\Delta\hat{G}_{(0,1)\to(0,1)}(x,t;\,x',t')\right\| \;\le \sum_{a\in\{\mathrm{spec},\mathrm{vac},\mathrm{corr}\}} \left\|\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}(x,t;\,x',t')\right\|. \tag{647}$$

For each triad label a, condition *(b)* of Definition 10.5 (triadic clustering, invoking Assumption 10.9) implies the existence of finite constants $\xi_a > 0$, $t_a > 0$ and amplitudes $\epsilon_{0,a} > 0$ such that, on the finite QEC patch $\mathcal{P} \subset \mathfrak{B}$ and for all t, t' in the cycle time window under consideration,

$$\left\|\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}(x,t;\,x',t')\right\| \;\le\; \epsilon_{0,a}\,\exp\!\Big(-\frac{d_a(x,x')}{\xi_a} - \frac{|t-t'|}{t_a}\Big), \qquad x,x' \in \mathcal{P}, \tag{648}$$

where d_a is the corresponding triadic distance. On the finite patch $\mathcal{P}$, any two reasonable distances are equivalent, so we may choose a single "correlation distance" d_{corr} (for instance the relational distance induced by $\bar{\Sigma}_{\mu\nu}$) and absorb the associated equivalence constants into the amplitudes $\epsilon_{0,a}$. Thus, after this redefinition, we may assume that

$$\left\|\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}(x,t;\,x',t')\right\| \;\le\; \epsilon_{0,a}\,\exp\!\Big(-\frac{d_{\mathrm{corr}}(x,x')}{\xi_a} - \frac{|t-t'|}{t_a}\Big), \qquad x,x' \in \mathcal{P}. \tag{649}$$

Define coarse–grained correlation parameters by

$$\xi := \max_{a\in\{\mathrm{spec},\mathrm{vac},\mathrm{corr}\}} \xi_a, \qquad t_{\mathrm{corr}} := \max_{a\in\{\mathrm{spec},\mathrm{vac},\mathrm{corr}\}} t_a, \qquad \epsilon_0 := \sum_{a\in\{\mathrm{spec},\mathrm{vac},\mathrm{corr}\}} \epsilon_{0,a}.$$

Since $\xi \ge \xi_a$ and $t_{\mathrm{corr}} \ge t_a$ for each a, we have

$$-\frac{d_{\mathrm{corr}}(x,x')}{\xi_a} - \frac{|t-t'|}{t_a} \;\le\; -\frac{d_{\mathrm{corr}}(x,x')}{\xi} - \frac{|t-t'|}{t_{\mathrm{corr}}},$$

and therefore

$$\exp\!\Big(-\frac{d_{\mathrm{corr}}(x,x')}{\xi_a} - \frac{|t-t'|}{t_a}\Big) \;\le\; \exp\!\Big(-\frac{d_{\mathrm{corr}}(x,x')}{\xi} - \frac{|t-t'|}{t_{\mathrm{corr}}}\Big). \tag{650}$$

Combining this with the previous bounds yields

$$\left\|\Delta\hat{G}_{(0,1)\to(0,1)}(x,t;\,x',t')\right\| \le \sum_a \epsilon_{0,a}\,\exp\!\Big(-\frac{d_{\mathrm{corr}}(x,x')}{\xi} - \frac{|t-t'|}{t_{\mathrm{corr}}}\Big) \tag{651}$$

$$= \epsilon_0\,\exp\!\Big(-\frac{d_{\mathrm{corr}}(x,x')}{\xi} - \frac{|t-t'|}{t_{\mathrm{corr}}}\Big), \tag{652}$$

for all $x, x' \in \mathcal{P}$ and all t, t' in the QEC cycle. This is precisely the clustered Twofold–block bound required in Assumption 4.5 for the correlation–only IQR noise model, with parameters $(\xi, t_{\mathrm{corr}}, \epsilon_0)$.

In particular, any Triadic–Fourfold noise kernel $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ induces, upon forgetting triad labels and the (p,s) sector, a correlation–only noise kernel in the sense of Assumption 4.5. Thus $\mathcal{N}_{\text{tri-4F}}$ is a *strict strengthening* of the earlier correlation–only IQR noise class. □

Remark 10.7 (Relation to the earlier correlation–only IQR noise class)**.** In the special case where vacuum and spectral contributions are negligible so that the only nontrivial triad label is $a = \mathrm{corr}$, and where the (p, s) sector is ignored, the Triadic–Fourfold Noise Class of Definition 10.5 reduces, via Lemma 10.6, to the correlation–only IQR noise class of Assumption 4.5. In this *correlation–only regime* there is a single correlation distance d_{corr} and correlation time t_{corr}, the Twofold–preserving block $\Delta\hat{G}_{(0,1)\to(0,1)}$ obeys the clustered bound with amplitude ϵ_0, and the cross blocks between the Twofold and (p, s) sectors collapse to a single effective leakage parameter. Thus, all noise data may be summarised by the triple $(d_{\mathrm{corr}}, t_{\mathrm{corr}}, \epsilon_0)$ and a single leakage rate, exactly as in the earlier correlation–only IQR formulation.

From the point of view of the present work, the Triadic–Fourfold Noise Class is therefore a *strict strengthening* of the correlation–only class: every $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ automatically satisfies the weaker correlation–only assumptions once triadic provenance (spec/vac/corr) and the (p, s) sector are coarse–grained away, so all results derived under Assumption 4.5 remain valid. The additional triadic tags then provide finer provenance bookkeeping within the full Tetralemma semantics (separating spectral, vacuum, and correlational channels), without requiring any change to the structure of the QEC threshold analysis itself.

10.3 Triadic clustering and QEC time window

We now introduce the triadic spatial and temporal scales which control the behaviour of the triad–resolved noise kernel, and we state the corresponding clustering and time–window assumptions in a form adapted to quantum error–correction, in line with the general IQR/PQS chronogeometric framework of [8] and standard clustering analyses for local quantum systems [53, 54].

Definition 10.8 (Triadic distances and times)**.** Let Assumptions 9.2 and 9.16 hold. A *triadic spatial distance system* on the PQS domain $\mathfrak{B} \subset \mathcal{X}_q$ consists of a triple of functions

$$d_a : \mathfrak{B} \times \mathfrak{B} \longrightarrow [0, \infty), \qquad a \in \{\mathrm{corr}, \mathrm{vac}, \mathrm{spec}\}, \tag{653}$$

such that for each label a the function d_a is a (possibly pseudo–)metric on $\mathfrak{B}$ that is compatible with the underlying chronogeometric structure induced by $\bar{\Sigma}_{\mu\nu}$ in the sense of the IQR chronogeometry [8].

A *triadic time system* consists of three positive real numbers

$$t_{\mathrm{corr}}, \qquad t_{\mathrm{vac}}, \qquad t_{\mathrm{spec}} \ > \ 0, \tag{654}$$

called respectively the *correlation time*, *vacuum time*, and *spectral time*. These are the characteristic time scales obtained from the time–triad construction of [8] (Chapters 5 and 15) and the Good–clock evolution U_t on $\mathcal{H}_{\mathrm{phys}}$, in analogy with correlation and relaxation times in open quantum system theory [9, 56]. For notational convenience we write

$$t_a := \begin{cases} t_{\mathrm{corr}}, & a = \mathrm{corr}, \\ t_{\mathrm{vac}}, & a = \mathrm{vac}, \\ t_{\mathrm{spec}}, & a = \mathrm{spec}. \end{cases} \tag{655}$$

In what follows we measure operator norms on $\mathcal{H}_{\text{phys}}$ using the usual operator norm $\|\cdot\|$ unless stated otherwise.

Assumption 10.9 (Triadic clustering (QC version))**.** *Let* $\Delta\hat{G}^{(a)}_{\bullet\to\bullet}(x,x')$ *denote any of the triad–resolved blocks of the noise kernel* $\Delta\hat{G}(x,x')$ *as in Definition 10.3. We assume that there exists a triadic spatial distance system* $\{d_a\}_{a\in\{\text{corr},\text{vac},\text{spec}\}}$ *on* $\mathfrak{B}$*, and positive constants* $\xi_a > 0$*,* $C_a > 0$ *for each* $a \in \{\text{corr}, \text{vac}, \text{spec}\}$*, such that:*

(i) Spatial clustering.
For all $x, x' \in \mathfrak{B}$ *and all triad labels* a*,*

$$\left\|\Delta\hat{G}^{(a)}_{\bullet\to\bullet}(x,x')\right\| \le C_a \exp\!\left(-\frac{d_a(x,x')}{\xi_a}\right), \tag{656}$$

expressing exponential clustering of the triad–resolved correlators, in the spirit of Lieb–Robinson–type bounds and cluster expansions for local quantum lattice systems [53, 54, 55].

(ii) Temporal clustering.
Let $t, t' \in \mathbb{R}$ *be Good–clock times associated with two QEC events supported at loci* $x, x' \in \mathfrak{B}$*, and let* $|t - t'|$ *denote their temporal separation. For each triad label* a *there exists a positive constant* C'_a *such that, for all* $|t - t'| \ge t_a$*,*

$$\left\|\Delta\hat{G}^{(a)}_{\bullet\to\bullet}(x,t;\, x',t')\right\| \le C'_a \exp\!\left(-\frac{|t-t'|}{t_a}\right), \tag{657}$$

where $\Delta\hat{G}^{(a)}_{\bullet\to\bullet}(x,t;\, x',t')$ *denotes the corresponding time–ordered, triad–resolved noise kernel in the Heisenberg picture with respect to the Good–clock evolution* U_t*. This is the analogue, in the IQR setting, of exponential decay of temporal correlations in mixing or quasi–Markovian environments [9, 56].*

Assumption 10.9 expresses the fact that, when resolved by triad provenance, the noise kernel decays exponentially both in spatial triadic distance and in temporal separation, with possibly different correlation lengths ξ_a and characteristic times t_a for the correlational, vacuum, and spectral channels. This is precisely the type of locality and clustering structure that underpins many rigorous fault–tolerance analyses for local noise models [2, 3, 5].

Definition 10.10 (Triadic QEC time window)**.** Let $C = (\hat{T}, U_t)$ be the Good–clock on $\mathcal{H}_{\text{phys}}$ appearing in Definition 9.1. A *QEC cycle* is specified by a Good–clock time interval $[t, t + \Delta t_{\text{QEC}}]$ in the spectrum of $\hat{T}$, during which a complete round of QEC operations (gates, syndrome extraction, decoding and correction) is implemented, in the sense of the QEC instrumentation $\mathscr{I}_{\text{QEC}}$ of Definition 9.1 and standard FTQC architectures [18, 17, 5].

We say that the QEC cycle satisfies the *triadic QEC time window* if the duration $\Delta t_{\text{QEC}} > 0$ obeys the strict scale separation

$$t_{\text{corr}} \ll \Delta t_{\text{QEC}} \ll t_{\text{vac}},\ t_{\text{spec}}, \tag{658}$$

where $t_{\text{corr}}, t_{\text{vac}}, t_{\text{spec}}$ are the triadic times of Definition 10.8. The symbols $\ll$ are to be understood in the sense that the ratios $\Delta t_{\text{QEC}}/t_{\text{vac}}$ and $\Delta t_{\text{QEC}}/t_{\text{spec}}$ are sufficiently small, and $t_{\mathrm{corr}}/\Delta t_{\mathrm{QEC}}$ is sufficiently small, to justify the approximations used in the effective noise model constructed below, mirroring the usual separation of correlation and control time scales in open–system models of quantum error–correction [5, 9].

Remark 10.11 (Interpretation of the triadic QEC time window)**.** The inequalities (658) have the following operational interpretation:

(a) The correlation time t_{corr} is much shorter than the QEC cycle duration Δt_{QEC}. This implies that the environment memory, as probed in the correlational channel, decays rapidly within a single cycle, and successive cycles can be treated, to a good approximation, as governed by a Markovian effective noise model, in line with the standard locality and weak–memory hypotheses of threshold theorems [2, 3, 5].

(b) The vacuum and spectral times t_{vac} and t_{spec} are much longer than Δt_{QEC}. Consequently, the vacuum and spectral structure seen by the QEC instrumentation is quasi–static over many cycles: the effective noise parameters derived from the vacuum and spectral triad blocks drift only slowly in Good–clock time, and can be treated as approximately constant over the time scales relevant for fault–tolerant operation, as in the time–triad and Good–clock construction of [8].

In this way, the Good–clock structure of [8] (especially Chapter 15) enters the QEC analysis explicitly: QEC cycles are chosen to sit in a time window where fast correlational fluctuations are averaged over, while vacuum and spectral backgrounds appear effectively frozen. This reconciles the IQR/Tetralemma chronogeometry with the usual discrete–time QEC picture used in surface–code and FTQC architectures [1, 4, 5].

11 Emergent register and triadic, sector–resolved leakage

In the correlation–only setting of Assumption 4.5, Phase 1 constructs the emergent Twofold register and a single leakage parameter $\varepsilon_{\mathrm{ps}}$, quantifying mixing between the Twofold and (p, s) sectors over one QEC cycle, within the general IQR/PQS framework of [8]. In this section we recall that emergent register construction and *refine* the leakage notion by resolving it along the triadic axes (correlational, vacuum, spectral) for noise models in the Triadic–Fourfold Noise Class $\mathcal{N}_{\text{tri-4F}}$, in line with leakage treatments in fault–tolerant schemes [17, 3, 5].

11.1 Emergent register and basic leakage norm

We begin by recalling the emergent register structure; the constructions and definitions in this subsection are unchanged from the correlation–only setting of Assumption 4.5 and follow the general pattern of emergent qubit registers in QEC and FTQC [13, 6, 5]. In the correlation–only analysis of Part A, the emergent register was embedded in a “total” Hilbert space $\mathcal{H}_{\mathrm{tot}}$; in the present QEC setting we identify this total Hilbert space with the BRST/plateau–cleaned physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$ of Definition 9.1.

Definition 11.1 (Emergent Twofold register)**.** Let $\mathcal{P}_L = \{x_1, \ldots, x_{n_L}\} \subset \mathfrak{B}$ be a finite configuration of PQS loci (the *QEC patch* of linear size L), and let $\mathcal{H}_{(0,1),x_j}$ be the Twofold subspace at x_j as in Definition 10.1. The *emergent register* of size L is the Hilbert space

$$\mathcal{H}_{\mathrm{reg}}^{(L)} := \bigotimes_{j=1}^{n_L} \mathcal{H}_{(0,1),x_j}, \tag{659}$$

which we identify with a qubit register $(\mathbb{C}^2)^{\otimes n_L}$ via the chosen Twofold bases $\{|0\rangle_{x_j}, |1\rangle_{x_j}\}$. The complementary subspace $\mathcal{H}_{\mathrm{comp}}^{(L)}$ is defined by a decomposition

$$\mathcal{H}_{\mathrm{phys}} \supset \mathcal{H}_{\mathrm{reg}}^{(L)} \oplus \mathcal{H}_{\mathrm{comp}}^{(L)}, \tag{660}$$

where $\mathcal{H}_{\mathrm{comp}}^{(L)}$ collects all degrees of freedom not belonging to the emergent register (including any parasymmetric/holo (p, s) components and any additional nonregister PQS modes). This is the same emergent register construction as in the correlation–only setting of Definition 5.2, now understood on the band/BRST–cleaned physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$ of Definition 9.1.

Definition 11.2 (Register and complement projectors)**.** Let $\mathcal{H}_{\mathrm{reg}}^{(L)}$ be as in Definition 11.1. We denote by

$$P_{\mathrm{reg}}^{(L)} : \mathcal{H}_{\mathrm{phys}} \to \mathcal{H}_{\mathrm{phys}} \tag{661}$$

the orthogonal projector onto $\mathcal{H}_{\mathrm{reg}}^{(L)}$, and by

$$P_{\mathrm{comp}}^{(L)} := \mathbf{1}_{\mathcal{H}_{\mathrm{phys}}} - P_{\mathrm{reg}}^{(L)} \tag{662}$$

the projector onto the complementary subspace $\mathcal{H}_{\mathrm{comp}}^{(L)}$. When the code size L is fixed and clear from context, we abbreviate these as P_{reg} and P_{comp}.

Definition 11.3 (Basic leakage norm)**.** Let $\mathcal{N} : \mathcal{B}(\mathcal{H}_{\mathrm{phys}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ be the CPTP map representing one full noisy QEC cycle on the physical Hilbert space. The associated *confined register channel* $\mathcal{N}_{\mathrm{reg}} : \mathcal{S}(\mathcal{H}_{\mathrm{reg}}^{(L)}) \to \mathcal{S}(\mathcal{H}_{\mathrm{reg}}^{(L)})$ is the CPTP map obtained by restricting $\mathcal{N}$ to the register sector and discarding any excursions into $\mathcal{H}_{\mathrm{comp}}^{(L)}$ (e.g. via the block–Kraus construction of Definition 3.15 and Lemma 3.18 in the correlation–only setting).

Equivalently, and for later use, we fix a Stinespring dilation of $\mathcal{N}$: there exist an auxiliary environment Hilbert space $\mathcal{H}_{\mathrm{env}}$, a reference environment state $\omega_{\mathrm{env}} \in \mathcal{S}(\mathcal{H}_{\mathrm{env}})$, and a unitary $U : \mathcal{H}_{\mathrm{phys}} \otimes \mathcal{H}_{\mathrm{env}} \to \mathcal{H}_{\mathrm{phys}} \otimes \mathcal{H}_{\mathrm{env}}$ such that

$$\mathcal{N}(\rho) = \mathrm{Tr}_{\mathrm{env}}\Big(U\,(\rho \otimes \omega_{\mathrm{env}})\,U^\dagger\Big), \qquad \rho \in \mathcal{S}(\mathcal{H}_{\mathrm{phys}}), \tag{663}$$

where $\mathrm{Tr}_{\mathrm{env}}$ denotes the partial trace over $\mathcal{H}_{\mathrm{env}}$.

The *leakage parameter* $\varepsilon_{\mathrm{ps}}$ is defined by

$$\varepsilon_{\mathrm{ps}} := \sup_{\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{reg}}^{(L)})} \Big\| \mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}_{\mathrm{reg}}(\rho) \Big\|_1, \tag{664}$$

where $\mathrm{Tr}_{\mathrm{comp}}$ denotes the partial trace over $\mathcal{H}_{\mathrm{comp}}^{(L)}$, so that $\mathrm{Tr}_{\mathrm{comp}}(\mathcal{N}(\rho))$ is the reduced state on $\mathcal{H}_{\mathrm{reg}}^{(L)}$ obtained from the full dynamics $\mathcal{N}$. The quantity $\varepsilon_{\mathrm{ps}}$ measures, in trace norm, the maximal deviation between the actual reduced register dynamics and the ideal confined register channel $\mathcal{N}_{\mathrm{reg}}$. In particular, $\varepsilon_{\mathrm{ps}} = 0$ if and only if there is no mixing between the register and its complement during one QEC cycle, in direct analogy with leakage/error parameters used in standard FTQC analyses [17, 3, 5].

In the Triadic–Fourfold Noise Class, the quantity $\varepsilon_{\mathrm{ps}}$ remains the primary leakage parameter entering the threshold theorem. The triadic refinement introduced below will allow us to attribute contributions to $\varepsilon_{\mathrm{ps}}$ to distinct physical mechanisms (correlational, vacuum, spectral, and Twofold $\leftrightarrow (p, s)$ mixing), without changing its basic definition, much as different microscopic error channels are grouped into a single effective physical error rate in conventional threshold theorems [2, 5].

11.2 Triad–resolved leakage parameters

We now exploit the triad projectors and the block structure of the noise kernel $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ to define triad–resolved leakage parameters and to distinguish between different physical origins of leakage, paralleling standard decompositions of noise into independent physical mechanisms in open quantum systems [9, 10].

Recall from Definition 10.3 that the cross blocks of the noise kernel between the Twofold and (p, s) sectors admit triad decompositions of the form

$$\Delta\hat{G}_{(0,1)\to(p,s)}(x, x') \;:=\; \sum_{a\in\{\mathrm{corr},\mathrm{vac},\mathrm{spec}\}} \Delta\hat{G}^{(a)}_{(0,1)\to(p,s)}(x, x'), \tag{665}$$

and similarly for $(p, s) \to (0, 1)$. For $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$, Definition 10.5 guarantees the existence of norm bounds on these blocks.

Definition 11.4 (Triad–resolved leakage parameters)**.** Let $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ be the noise kernel for one QEC cycle on a patch $\mathcal{P}_L$ and let $\mathcal{N} : \mathcal{B}(\mathcal{H}_{\mathrm{phys}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ be the corresponding CPTP map, with fixed Stinespring dilation $(\mathcal{H}_{\mathrm{env}}, \omega_{\mathrm{env}}, U)$ as in Definition 11.3. For each triad label $a \in \{\mathrm{corr}, \mathrm{vac}, \mathrm{spec}\}$ we define the *triad–resolved leakage parameter* $\varepsilon^{(a)}_{\mathrm{ps}}$ to be any nonnegative number satisfying both:

(a) the cross–block bounds in correlator space

$$\sup_{x,x'\in\mathcal{P}_L} \Big(\Big\| \Delta\hat{G}^{(a)}_{(0,1)\to(p,s)}(x, x') \Big\| + \Big\| \Delta\hat{G}^{(a)}_{(p,s)\to(0,1)}(x, x') \Big\| \Big) \;\le\; \varepsilon^{(a)}_{\mathrm{ps}}, \tag{666}$$

as in Definition 10.5;

(b) the corresponding channel–level leakage bound

$$\sup_{\rho\in\mathcal{S}(\mathcal{H}^{(L)}_{\mathrm{reg}})} \Big\| \mathrm{Tr}_{\mathrm{comp}}\, \mathrm{Tr}_{\mathrm{env}} \Big(\mathcal{N}^{(a)}(\rho \otimes \omega_{\mathrm{env}}) \Big) - \mathcal{N}^{(a)}_{\mathrm{reg}}(\rho) \Big\|_1 \;\le\; C_a\, \varepsilon^{(a)}_{\mathrm{ps}}, \tag{667}$$

for some constant $C_a \ge 1$ depending only on the size of the patch and on the details of the reconstruction of $\mathcal{N}$ from $\Delta\hat{G}$. Here $\mathcal{N}^{(a)}$ denotes the contribution to the *dilated* channel on $\mathcal{B}(\mathcal{H}_{\mathrm{phys}} \otimes \mathcal{H}_{\mathrm{env}})$ induced by the triad–resolved cross blocks $\Delta\hat{G}^{(a)}_{(0,1)\leftrightarrow(p,s)}$, and

$\mathcal{N}_{\mathrm{reg}}^{(a)}$ is the corresponding confined register channel obtained by applying the procedure of Definition 11.3 to the reduced system channel

$$\rho \longmapsto \operatorname{Tr}_{\mathrm{env}}\Big(\mathcal{N}^{(a)}(\rho \otimes \omega_{\mathrm{env}})\Big),$$

in analogy with channel reconstruction from correlators via Choi matrices [57, 7].

We then define the *total triad–resolved leakage* as

$$\varepsilon_{\mathrm{ps}}^{\mathrm{triad}} := \varepsilon_{\mathrm{ps}}^{(\mathrm{corr})} + \varepsilon_{\mathrm{ps}}^{(\mathrm{vac})} + \varepsilon_{\mathrm{ps}}^{(\mathrm{spec})}. \tag{668}$$

The next lemma records the relation between the basic leakage norm $\varepsilon_{\mathrm{ps}}$ and its triadic refinements.

Lemma 11.5 (Basic leakage vs. triad–resolved leakage)**.** *Let $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ and let $\mathcal{N}$ be the corresponding one–cycle channel on $\mathcal{H}_{\mathrm{phys}}$, with fixed Stinespring dilation $(\mathcal{H}_{\mathrm{env}}, \omega_{\mathrm{env}}, U)$ as in Definition 11.3. Then the basic leakage parameter $\varepsilon_{\mathrm{ps}}$ defined in (664) obeys the bound*

$$\varepsilon_{\mathrm{ps}} \le C\left(\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})} + \varepsilon_{\mathrm{ps}}^{(\mathrm{vac})} + \varepsilon_{\mathrm{ps}}^{(\mathrm{spec})}\right) = C\,\varepsilon_{\mathrm{ps}}^{\mathrm{triad}}, \tag{669}$$

for some constant $C \ge 1$ depending only on the patch size and on the channel reconstruction scheme.

Proof. By Definition 11.3, the basic leakage parameter is

$$\varepsilon_{\mathrm{ps}} = \sup_{\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{reg}}^{(L)})} \left\| \operatorname{Tr}_{\mathrm{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}_{\mathrm{reg}}(\rho) \right\|_1. \tag{670}$$

Using the fixed Stinespring dilation, we may equivalently write

$$\mathcal{N}(\rho) = \operatorname{Tr}_{\mathrm{env}}\Big(\mathcal{N}^{\mathrm{dil}}(\rho \otimes \omega_{\mathrm{env}})\Big), \tag{671}$$

for a suitable dilated channel $\mathcal{N}^{\mathrm{dil}} : \mathcal{B}(\mathcal{H}_{\mathrm{phys}} \otimes \mathcal{H}_{\mathrm{env}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}} \otimes \mathcal{H}_{\mathrm{env}})$ generated by the unitary U. By linearity of the reconstruction of $\mathcal{N}^{\mathrm{dil}}$ from the noise kernel $\Delta\hat{G}$, the contribution of the cross blocks to $\mathcal{N}^{\mathrm{dil}}$ decomposes as a sum of the triad–resolved contributions $\mathcal{N}^{(a)}$, and similarly for the register restriction:

$$\mathcal{N}^{\mathrm{dil}} = \mathcal{N}^{\mathrm{diag}} + \sum_{a \in \{\mathrm{corr},\mathrm{vac},\mathrm{spec}\}} \mathcal{N}^{(a)}, \tag{672}$$

where $\mathcal{N}^{\mathrm{diag}}$ collects the block–preserving (register ↔ register and complement ↔ complement) contributions. By construction of the confined register channel, the block–preserving part does not contribute to leakage, so for any register state ρ we have

$$\operatorname{Tr}_{\mathrm{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}_{\mathrm{reg}}(\rho) = \sum_{a \in \{\mathrm{corr},\mathrm{vac},\mathrm{spec}\}} \Big[\operatorname{Tr}_{\mathrm{comp}} \operatorname{Tr}_{\mathrm{env}}\Big(\mathcal{N}^{(a)}(\rho \otimes \omega_{\mathrm{env}})\Big) - \mathcal{N}_{\mathrm{reg}}^{(a)}(\rho)\Big]. \tag{673}$$

Taking the trace norm and using the triangle inequality, followed by the channel–level bounds (667), yields

$$\left\|\mathrm{Tr}_{\mathrm{comp}}\Big(\mathcal{N}(\rho)\Big) - \mathcal{N}_{\mathrm{reg}}(\rho)\right\|_1 = \left\|\mathrm{Tr}_{\mathrm{comp}}\,\mathrm{Tr}_{\mathrm{env}}\Big(\mathcal{N}^{\mathrm{dil}}(\rho\otimes\omega_{\mathrm{env}})\Big) - \mathcal{N}_{\mathrm{reg}}(\rho)\right\|_1 \tag{674}$$

$$\leq \sum_a \left\|\mathrm{Tr}_{\mathrm{comp}}\,\mathrm{Tr}_{\mathrm{env}}\Big(\mathcal{N}^{(a)}(\rho\otimes\omega_{\mathrm{env}})\Big) - \mathcal{N}^{(a)}_{\mathrm{reg}}(\rho)\right\|_1 \tag{675}$$

$$\leq \sum_a C_a\,\varepsilon^{(a)}_{\mathrm{ps}} \;\leq\; C\,\varepsilon^{\mathrm{triad}}_{\mathrm{ps}}, \tag{676}$$

where $C := \max_a C_a$. Taking the supremum over all $\rho \in \mathcal{S}(\mathcal{H}^{(L)}_{\mathrm{reg}})$ then gives (669). □

Remark 11.6 (Physical interpretation: dangerous vs. slow leakage)**.** The triad–resolved leakage parameters admit the following qualitative interpretation in the context of fault–tolerant quantum computation:

- $\varepsilon^{(\mathrm{corr})}_{\mathrm{ps}}$ captures leakage mediated by the *correlational* channel on the shortest time scale t_{corr}. These processes correspond to fast couplings between the emergent Twofold register and its complement, on time scales comparable to or shorter than a single QEC cycle. Operationally they behave like conventional, rapidly fluctuating leakage errors and directly contribute to the "dangerous" part of $\varepsilon_{\mathrm{ps}}$ that must be kept below the threshold, in line with local, weakly correlated noise models in rigorous FTQC analyses [2, 3, 5].
- $\varepsilon^{(\mathrm{vac})}_{\mathrm{ps}}$ and $\varepsilon^{(\mathrm{spec})}_{\mathrm{ps}}$ quantify leakage associated with the *vacuum* and *spectral* channels on the much longer time scales t_{vac} and t_{spec}. In the triadic QEC time window of Definition 10.10,

  $$t_{\mathrm{corr}} \;\ll\; \Delta t_{\mathrm{QEC}} \;\ll\; t_{\mathrm{vac}},\, t_{\mathrm{spec}},$$

  these contributions are quasi–static over many QEC cycles. Physically, they often manifest as slow drift of background structures (e.g. chronogeometric or vacuum dressing) or of effective spectral parameters, leading to a gradual renormalisation of gate generators and measurement settings rather than to rapidly fluctuating leakage events. To leading order, such slow components can be absorbed into a redefinition of the "ideal" dynamics (and hence of $\hat{G}^0_\Lambda$) and tracked as adiabatic drift, in the spirit of slowly varying noise models in open quantum systems [9, 10].

In the threshold analysis, the basic leakage parameter $\varepsilon_{\mathrm{ps}}$ from (664) remains the primary quantity entering the bounds. However, Definitions 11.4 and Lemma 11.5 show that $\varepsilon_{\mathrm{ps}}$ can be decomposed into triadic contributions $\varepsilon^{(a)}_{\mathrm{ps}}$ tied to their characteristic scales (t_a, ξ_a). This clarifies which parts of the leakage budget are genuinely dangerous for fault–tolerant operation (fast, correlational leakage) and which are better viewed as slowly varying background dressing that can be folded into the IQR notion of an ideal QEC cycle and monitored perturbatively.

12 Effective local noise parameters from triadic–fourfold data

In the correlation–only setting of Assumption 4.5, Phase 2 reduces the physical noise for one QEC cycle to a local Pauli noise model on the emergent register, characterised by:

- a per–site physical error rate p,
- a process differential parameter δ controlling the difference between two syndrome–extraction schedules,
- and a leakage parameter $\varepsilon_{\mathrm{ps}}$.

In the Triadic–Fourfold Noise Class $\mathcal{N}_{\text{tri-4F}}$, these quantities can be derived more explicitly as functionals of the triad– resolved blocks of the noise kernel $\Delta\hat{G}$. The present section formalises this derivation at the level needed for the QEC threshold theorem, in direct analogy with standard local–noise reductions in rigorous threshold analyses [2, 3, 16, 5].

12.1 Twofold–preserving block and triad–split error rates

We first recall the construction of the reduced single–site channel on the Twofold sector and then define triad–split error rates associated with the Twofold–preserving block of the noise kernel, following the general Choi/diamond–norm channel comparison framework [57, 7].

Definition 12.1 (Single–site reduced channel)**.** Let $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ be the noise kernel for one QEC cycle on a patch $\mathcal{P}_L = \{x_1, \ldots, x_{n_L}\}$, and let $\mathcal{N} : \mathcal{B}(\mathcal{H}_{\mathrm{phys}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ be the corresponding reduced CPTP map for that cycle (obtained from a fixed Stinespring dilation by tracing out the external environment, as in Definition 11.3). For each site index j, we define the *single–site reduced channel* $\mathcal{N}_j^{(1)} : \mathcal{S}(\mathcal{H}_{(0,1),x_j}) \to \mathcal{S}(\mathcal{H}_{(0,1),x_j})$ by:

(a) embedding $\rho_j \in \mathcal{S}(\mathcal{H}_{(0,1),x_j})$ into the full physical space as $\rho_j \otimes \omega_{\mathrm{rest}}$, where ω_{rest} is a fixed reference state on the remaining degrees of freedom within $\mathcal{H}_{\mathrm{phys}}$ (other Twofold fibres in $\mathcal{P}_L \setminus \{x_j\}$, ancilla modes, and complementary PQS degrees not belonging to the emergent register);

(b) applying $\mathcal{N}$ to obtain $\mathcal{N}(\rho_j \otimes \omega_{\mathrm{rest}})$;

(c) projecting onto the Twofold sector at x_j and tracing out all other degrees of freedom in $\mathcal{H}_{\mathrm{phys}}$, i.e.

$$\mathcal{N}_j^{(1)}(\rho_j) := \mathrm{Tr}_{\mathrm{rest,comp}}\Big[P_{(0,1),x_j}\, \mathcal{N}(\rho_j \otimes \omega_{\mathrm{rest}})\, P_{(0,1),x_j}\Big], \tag{677}$$

where $P_{(0,1),x_j}$ is the projector onto $\mathcal{H}_{(0,1),x_j}$ inside the emergent register, and $\mathrm{Tr}_{\mathrm{rest,comp}}$ denotes the partial trace over all factors in $\mathcal{H}_{\mathrm{phys}}$ other than $\mathcal{H}_{(0,1),x_j}$.

The resulting map $\mathcal{N}_j^{(1)}$ is completely positive and trace–preserving on $\mathcal{S}(\mathcal{H}_{(0,1),x_j})$.

To isolate the contribution of each triad axis to the local noise, we now use the triad–resolved Twofold–preserving blocks $\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}$.

Definition 12.2 (Triad–split single–site error rates)**.** Let $\mathcal{N}_j^{(1)}$ be as in Definition 12.1 and let $\mathcal{N}_j^{(1),0}$ denote the corresponding *ideal* single–site channel (obtained from the ideal correlator $\hat{G}^0_\Lambda$ by the same construction). Because the reconstruction of $\mathcal{N}_j^{(1)}$ from $\Delta\hat{G}$ is linear, the deviation from the ideal channel decomposes as

$$\mathcal{N}_j^{(1)} - \mathcal{N}_j^{(1),0} \;=\; \sum_{a\in\{\mathrm{corr,vac,spec}\}} \Delta\mathcal{N}_j^{(1,a)}, \tag{678}$$

where each $\Delta\mathcal{N}_j^{(1,a)}$ is the contribution obtained from the triad–resolved Twofold–preserving block $\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}$ alone.

For each triad label $a \in \{\mathrm{corr}, \mathrm{vac}, \mathrm{spec}\}$ we define the *triad–split single–site error rate* at site x_j by

$$p_a^{(j)} \;:=\; \frac{1}{2} \sup_{\rho_j \in \mathcal{S}(\mathcal{H}_{(0,1),x_j})} \left\| \Delta\mathcal{N}_j^{(1,a)}(\rho_j) \right\|_1. \tag{679}$$

The *total single–site error rate* at x_j is then

$$p_{\mathrm{tot}}^{(j)} \;:=\; p_{\mathrm{corr}}^{(j)} + p_{\mathrm{vac}}^{(j)} + p_{\mathrm{spec}}^{(j)}. \tag{680}$$

For a given patch $\mathcal{P}_L$, we define the corresponding global quantities

$$p_{\mathrm{corr}} \;:=\; \max_j p_{\mathrm{corr}}^{(j)}, \qquad p_{\mathrm{vac}} \;:=\; \max_j p_{\mathrm{vac}}^{(j)}, \qquad p_{\mathrm{spec}} \;:=\; \max_j p_{\mathrm{spec}}^{(j)}, \tag{681}$$

and

$$p_{\mathrm{tot}} \;:=\; p_{\mathrm{corr}} + p_{\mathrm{vac}} + p_{\mathrm{spec}}. \tag{682}$$

The next lemma collects the existence of an effective single–qubit Pauli channel approximating $\mathcal{N}_j^{(1)}$, with a diamond–norm error controlled by the triad–split error rates and the triadic clustering scales, in line with Pauli–twirling and effective local noise constructions used in threshold proofs [2, 3, 16, 5].

Lemma 12.3 (Triadic IQR → effective Pauli noise)**.** *Let $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ and let $\mathcal{N}_j^{(1)}$ be the single–site reduced channel at x_j as in Definition 12.1. Then there exists a single–qubit Pauli channel $\Lambda_j : \mathcal{S}(\mathcal{H}_{(0,1),x_j}) \to \mathcal{S}(\mathcal{H}_{(0,1),x_j})$ of the form*

$$\Lambda_j(\rho) \;=\; \sum_{P \in \{\mathbb{I},X,Y,Z\}} q_P^{(j)}\, P\rho P, \qquad q_P^{(j)} \geq 0, \quad \sum_P q_P^{(j)} = 1, \tag{683}$$

such that

$$\left\| \mathcal{N}_j^{(1)} - \Lambda_j \right\|_\diamond \;\leq\; \epsilon_{\mathrm{loc}}^{(j)}, \tag{684}$$

where $\|\cdot\|_\diamond$ denotes the diamond norm, and $\epsilon_{\mathrm{loc}}^{(j)}$ is bounded by a function

$$\epsilon_{\mathrm{loc}}^{(j)} \;\leq\; f_{\mathrm{loc}}\Big(p_{\mathrm{corr}}^{(j)},\, p_{\mathrm{vac}}^{(j)},\, p_{\mathrm{spec}}^{(j)};\, \{\xi_a\},\, \{t_a\}\Big), \tag{685}$$

for some nondecreasing function f_{loc} of its arguments, which depends only on the triadic clustering lengths $\{\xi_a\}$ and times $\{t_a\}$ and on the reconstruction scheme relating $\Delta\hat{G}$ to $\mathcal{N}_j^{(1)}$. In particular, in the triadic QEC time window of Definition 10.10, the dominant contribution to $\epsilon_{\mathrm{loc}}^{(j)}$ arises from $p_{\mathrm{corr}}^{(j)}$, whereas the contributions from $p_{\mathrm{vac}}^{(j)}$ and $p_{\mathrm{spec}}^{(j)}$ are suppressed by the large vacuum and spectral time scales $t_{\mathrm{vac}}, t_{\mathrm{spec}}$ and the corresponding spatial clustering lengths $\xi_{\mathrm{vac}}, \xi_{\mathrm{spec}}$.

Proof sketch. By construction, $\mathcal{N} : \mathcal{B}(\mathcal{H}_{\mathrm{phys}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ is the reduced one–cycle channel obtained from a fixed Stinespring dilation by tracing out the external environment (as in Definition 11.3); all dependence on the environment reference state is absorbed into $\mathcal{N}$ and hence into the single–site reduced channels $\mathcal{N}_j^{(1)}$ of Definition 12.1.

Step 1: Channel decomposition. By Definition 12.2, the deviation of the single–site channel from its ideal decomposes as

$$\mathcal{N}_j^{(1)} - \mathcal{N}_j^{(1),0} \;=\; \sum_{a\in\{\mathrm{corr},\mathrm{vac},\mathrm{spec}\}} \Delta\mathcal{N}_j^{(1,a)}, \tag{686}$$

where each $\Delta\mathcal{N}_j^{(1,a)}$ is reconstructed linearly from the triad–resolved Twofold–preserving block $\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}$ via the usual Choi construction [57, 7]. For later convenience we write

$$\mathcal{N}_{j,a}^{(1)} \;:=\; \mathcal{N}_j^{(1),0} + \Delta\mathcal{N}_j^{(1,a)}, \qquad a \in \{\mathrm{corr},\mathrm{vac},\mathrm{spec}\}. \tag{687}$$

The strength of each contribution is quantified at the state level by the triad–split error rate $p_a^{(j)}$ in (679).

Step 2: Correlational sector and Pauli covariance. On the correlational sector $a = \mathrm{corr}$, the hypotheses of Assumption 4.13 apply to the Twofold–preserving block $\Delta\hat{G}^{(\mathrm{corr})}_{(0,1)\to(0,1)}$. The corresponding local channel $\mathcal{N}_{j,\mathrm{corr}}^{(1)} := \mathcal{N}_{j,a=\mathrm{corr}}^{(1)}$ satisfies approximate Pauli covariance in the sense required by Lemma 4.15. Hence there exists a single–qubit Pauli channel $\Lambda_{j,\mathrm{corr}}$ such that

$$\left\|\mathcal{N}_{j,\mathrm{corr}}^{(1)} - \Lambda_{j,\mathrm{corr}}\right\|_\diamond \;\leq\; \epsilon_{\mathrm{corr}}^{(j)}, \tag{688}$$

with $\epsilon_{\mathrm{corr}}^{(j)}$ controlled by the correlational Twofold block size and the twirling error, as in Corollary 4.20. On a single qubit, the diamond norm is equivalent (up to universal constants) to the induced trace–norm deviation on states [7], so $\epsilon_{\mathrm{corr}}^{(j)}$ can be bounded in terms of $p_{\mathrm{corr}}^{(j)}$.

Step 3: Vacuum and spectral sectors. The vacuum and spectral contributions $\Delta\mathcal{N}_j^{(1,\mathrm{vac})}$ and $\Delta\mathcal{N}_j^{(1,\mathrm{spec})}$ are generated from the corresponding blocks $\Delta\hat{G}^{(\mathrm{vac})}$ and $\Delta\hat{G}^{(\mathrm{spec})}$, which satisfy the triadic clustering bounds of Assumption 10.9 and the time–scale separation of Definition 10.10. In particular, the large time scales $t_{\mathrm{vac}}, t_{\mathrm{spec}}$ and the associated spatial lengths $\xi_{\mathrm{vac}}, \xi_{\mathrm{spec}}$ imply that the vacuum and spectral contributions vary slowly across the patch and across cycles, and enter $\mathcal{N}_j^{(1)}$ as small coherent or quasi–static distortions of the ideal dynamics.

Standard bounds relating channel deviations to their Choi matrices [57, 7], combined with the exponential clustering (656)–(657), imply that

$$\left\|\Delta\mathcal{N}_j^{(1,\mathrm{vac})}\right\|_\diamond \;\lesssim\; p_{\mathrm{vac}}^{(j)}\, c_{\mathrm{vac}}(\{\xi_a\},\{t_a\}), \qquad \left\|\Delta\mathcal{N}_j^{(1,\mathrm{spec})}\right\|_\diamond \;\lesssim\; p_{\mathrm{spec}}^{(j)}\, c_{\mathrm{spec}}(\{\xi_a\},\{t_a\}), \tag{689}$$

for some prefactors $c_{\mathrm{vac}}, c_{\mathrm{spec}}$ depending only on the clustering data and the reconstruction scheme. In the triadic QEC time window, these prefactors are parametrically suppressed by the ratios $\Delta t_{\mathrm{QEC}}/t_{\mathrm{vac}}$, $\Delta t_{\mathrm{QEC}}/t_{\mathrm{spec}}$ and by the large spatial clustering lengths.

Step 4: Combining sectors and defining Λ_j. Let Λ_j be the Pauli channel obtained from $\Lambda_{j,\mathrm{corr}}$ by absorbing the leading coherent corrections from the vacuum and spectral sectors into its parameters $\{q_P^{(j)}\}_{P\in\{\mathbb{I},X,Y,Z\}}$. Then we can write

$$\mathcal{N}_j^{(1)} - \Lambda_j = \left(\mathcal{N}_{j,\mathrm{corr}}^{(1)} - \Lambda_{j,\mathrm{corr}}\right) + \Delta\mathcal{N}_j^{(1,\mathrm{vac})} + \Delta\mathcal{N}_j^{(1,\mathrm{spec})} + (\text{small renormalisation terms}). \tag{690}$$

Applying the triangle inequality for the diamond norm and using the bounds from Steps 2 and 3 gives

$$\left\| \mathcal{N}_j^{(1)} - \Lambda_j \right\|_\diamond \leq f_{\mathrm{loc}}\Big(p_{\mathrm{corr}}^{(j)}, p_{\mathrm{vac}}^{(j)}, p_{\mathrm{spec}}^{(j)}; \{\xi_a\}, \{t_a\}\Big), \tag{691}$$

for some nondecreasing function f_{loc} depending only on the clustering data and the reconstruction scheme, which yields (684)–(685). The scale separation $t_{\mathrm{corr}} \ll \Delta t_{\mathrm{QEC}} \ll t_{\mathrm{vac}}, t_{\mathrm{spec}}$ then ensures that, at fixed $p_{\mathrm{corr}}^{(j)}$, the dependence of f_{loc} on $p_{\mathrm{vac}}^{(j)}$ and $p_{\mathrm{spec}}^{(j)}$ is parametrically suppressed, so the correlational error rate provides the leading contribution to $\epsilon_{\mathrm{loc}}^{(j)}$. ☐

Remark 12.4 (Interpretation of triad–split error rates)**.** By construction and in light of Lemma 12.3:

- p_{corr} captures the usual short–range, Pauli–type noise generated by fast correlational fluctuations of the environment on the time scale t_{corr} and length scale ξ_{corr}. This is the direct analogue of the local, weakly correlated noise that appears in standard QEC/FTQC models [2, 3, 5], and it provides the dominant contribution to the stochastic part of the effective Pauli channel in Lemma 12.3.

- p_{vac} quantifies errors arising from slow vacuum variations across the QEC patch and across cycles, e.g. gradual changes in the realized chronogeometry $\bar{\Sigma}_{\mu\nu}$ or in other background fields, governed by the long time scale t_{vac} and length scale ξ_{vac}. On QEC time scales selected by the triadic QEC time window, such effects typically manifest as slow drifts in effective Hamiltonians or gate parameters, in the spirit of slowly varying environments in open quantum systems [9, 10].

- p_{spec} measures errors due to spectral shear or band–edge misalignment, such as off–band excitations relative to the UV/IR filters $P_x(\lambda_{\mathrm{UV}})$ and $K_{\mathrm{IR}}^{(r)}$, i.e. imperfect confinement of the dynamics to the intended band–limited sector of the QEC–realized weave configuration (Definition 9.14). These errors are controlled by the spectral time t_{spec} and length ξ_{spec}.

The triadic QEC time window (658) ensures that the contributions associated with p_{vac} and p_{spec} typically vary only slowly over many cycles and may, to leading order, be absorbed into effective Hamiltonian renormalisations or small corrections to gate parameters. By contrast, p_{corr} controls the fast, cycle–to–cycle fluctuating component of the noise and hence dominates the stochastic error budget that is relevant for threshold behaviour, in direct analogy with the local noise assumptions used in standard threshold theorems [2, 3, 5]. The triad–split error rates therefore disentangle the physically distinct sources of local noise while leaving the overall threshold analysis in the same structural form as in the correlation–only setting.

12.2 Process differential noise and triad–resolved overlaps

We now turn to the process–level differential noise between two syndrome extraction schedules, which will enter the holosymmetry bounds. In the correlation–only setting, a single Bhattacharyya overlap parameter k and its complement $\delta = 1 - k$ were used to quantify process proximity at the Pauli level, in line with standard information–theoretic noise metrics for quantum channels [6, 7, 8]. Here we refine this by triad provenance.

Definition 12.5 (Per–site Pauli distributions for two schedules)**.** Let A and B be two QEC schedules (for example, two different implementations of one full round of stabilizer extraction and correction) on the same patch $\mathcal{P}_L$, both realised on the ambient configuration $\mathscr{D}_{\mathrm{QEC}}$ and weave $\mathcal{I}^{[\mathscr{S}]}$. Let $\mathcal{N}^{(1)}_{j,A}$ and $\mathcal{N}^{(1)}_{j,B}$ be the corresponding single–site reduced channels at x_j (as in Definition 12.1), and let $\Lambda_{j,A}, \Lambda_{j,B}$ be their effective Pauli approximations as in Lemma 12.3.[3] We write

$$\Lambda_{j,\alpha}(\rho) \;=\; \sum_{P\in\{\mathbb{I},X,Y,Z\}} q_P^{(j,\alpha)}\, P\rho P, \qquad \alpha\in\{A,B\}, \tag{692}$$

and refer to $\mathbf{q}^{(j,\alpha)} = \{q_P^{(j,\alpha)}\}_P$ as the *per–site Pauli error distribution* of schedule α at x_j. This is the usual local Pauli noise description employed in rigorous threshold analyses for stabilizer codes and topological QEC schemes [13, 1, 2, 3, 5].

The Triadic–Fourfold structure allows us to further decompose these distributions according to triad provenance.

Definition 12.6 (Triad–resolved Pauli distributions and overlaps)**.** Under the hypotheses of Definition 12.5, suppose that for each triad label $a \in \{\mathrm{corr}, \mathrm{vac}, \mathrm{spec}\}$ we can identify nonnegative numbers $q_P^{(j,\alpha,a)}$ such that

$$q_P^{(j,\alpha)} \;=\; q_P^{(j,\alpha,\mathrm{corr})} + q_P^{(j,\alpha,\mathrm{vac})} + q_P^{(j,\alpha,\mathrm{spec})}, \qquad \alpha\in\{A,B\}, \tag{693}$$

with $\sum_P q_P^{(j,\alpha,a)} \le 1$ for each fixed triad label a, and $\sum_a \sum_P q_P^{(j,\alpha,a)} = 1$ for each fixed (j,α). Operationally, the $q_P^{(j,\alpha,a)}$ arise from Pauli approximations to the triad–resolved contributions $\mathcal{N}^{(1,a)}_{j,\alpha}$ of the single–site channel $\mathcal{N}^{(1)}_{j,\alpha}$, reconstructed from the triad–resolved Twofold–preserving blocks $\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}$ via the Choi construction [57, 7, 8].

We then define, for each triad label a, the *triad–resolved Bhattacharyya overlap* at site x_j by

$$k_a^{(j)} \;:=\; \sum_{P\in\{\mathbb{I},X,Y,Z\}} \sqrt{q_P^{(j,A,a)}\, q_P^{(j,B,a)}}, \qquad a\in\{\mathrm{corr},\mathrm{vac},\mathrm{spec}\}, \tag{694}$$

and the corresponding *triad–resolved process differential* by

$$\delta_a^{(j)} \;:=\; 1 - k_a^{(j)}. \tag{695}$$

The total per–site overlap and differential are defined as

$$k_{\mathrm{tot}}^{(j)} \;:=\; \sum_P \sqrt{q_P^{(j,A)}\, q_P^{(j,B)}}, \qquad \delta_{\mathrm{tot}}^{(j)} \;:=\; 1 - k_{\mathrm{tot}}^{(j)}, \tag{696}$$

and we define global quantities

$$\delta_a \;:=\; \max_j \delta_a^{(j)}, \qquad \delta_{\mathrm{tot}} \;:=\; \max_j \delta_{\mathrm{tot}}^{(j)}. \tag{697}$$

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[3] Lemma 12.3 is applied separately to schedules A and B for each site x_j.

This triad–resolved decomposition mirrors the splitting of microscopic noise mechanisms into distinct physical channels (correlational, vacuum, spectral) familiar from open quantum systems and non–equilibrium environments [9, 10, 8], now lifted to the level of process overlaps between QEC schedules.

Remark 12.7 (Relation between total and triad–resolved differentials)**.** In general, δ_{tot} and the triad–resolved differentials δ_a are related in a nonlinear manner, because the Bhattacharyya overlap of a sum of contributions does not simply decompose into a sum of overlaps. However, under mild factorisation assumptions on the triad–resolved contributions at a fixed site x_j (for example, that $\mathcal{N}^{(1)}_{j,\alpha}$ can be approximated as a composition of approximately independent Pauli channels associated with the correlational, vacuum, and spectral sectors, which commute and whose error probabilities are small), one expects an approximate factorisation

$$k^{(j)}_{\text{tot}} \;\approx\; k^{(j)}_{\text{corr}}\, k^{(j)}_{\text{vac}}\, k^{(j)}_{\text{spec}}. \tag{698}$$

In that regime,

$$\delta^{(j)}_{\text{tot}} \;=\; 1 - k^{(j)}_{\text{tot}} \;\approx\; \delta^{(j)}_{\text{corr}} + \delta^{(j)}_{\text{vac}} + \delta^{(j)}_{\text{spec}}, \tag{699}$$

up to higher–order corrections in the small quantities $\delta^{(j)}_a$. Because schedules A and B are constructed on the same QEC diagnostic realized reference configuration and share the same Good–clock and band–limiting, the vacuum and spectral contributions are expected to be very similar between A and B, so that δ_{vac} and δ_{spec} are typically much smaller than δ_{corr}. In this regime one has, to leading order,

$$\delta_{\text{tot}} \;\approx\; \delta_{\text{corr}}, \tag{700}$$

and it is δ_{tot} (or equivalently δ_{corr} in this approximate sense) that enters the holosymmetry–based process threshold conditions in the subsequent analysis, in direct analogy with how dominant local noise parameters control process–level thresholds in standard fault–tolerant frameworks [1, 2, 3, 5, 8]. Importantly, the threshold proofs below only require small δ_{tot} as defined in (696)–(697); the approximate identification $\delta_{\text{tot}} \approx \delta_{\text{corr}}$ serves as a physical interpretation rather than an additional hypothesis.

13 Triadic–aware threshold: modified Phases 3–5

In this section we incorporate the triadic decomposition of the noise kernel into the threshold analysis, while leaving the QEC architecture itself (surface–code family, emergent register, and stabilizer structure) unchanged. The only modification relative to the correlation–only setting of Assumption 4.5 is that the effective local channel parameters are now expressed in terms of the triad–split quantities $(p_{\text{corr}}, p_{\text{vac}}, p_{\text{spec}})$ and the triad–resolved process differential δ_{tot}, in a way that fits into the standard surface–code threshold framework for local noise models [1, 11, 12, 4, 2, 3, 5] and the IQR/PQS setting of [8].

13.1 Surface–code family and stabilizer robustness

We first recall the code family and stabilizer robustness functional from the correlation–only setting, noting that their definitions do not depend on the triadic structure and follow the usual planar/surface–code constructions in the literature [1, 11, 12, 4, 14, 15, 5].

Definition 13.1 (Surface–code family on the emergent register)**.** Let $\{\mathcal{P}_L\}_{L\geq 1}$ be a family of finite PQS patches $\mathcal{P}_L \subset \mathfrak{B}$ of linear size L, and let $\mathcal{H}^{(L)}_{\mathrm{reg}}$ be the corresponding emergent register as in Definition 11.1. A *surface–code family* $\{\mathcal{C}_L\}_{L\geq 1}$ on $\{\mathcal{H}^{(L)}_{\mathrm{reg}}\}$ is a collection of code spaces $\mathcal{C}_L \subset \mathcal{H}^{(L)}_{\mathrm{reg}}$ with the following properties:

(a) Each $\mathcal{C}_L$ is the $+1$ eigenspace of an abelian stabilizer group $S_L \subset \mathcal{P}_{n_L}$ acting on $n_L = |\mathcal{P}_L|$ qubits, with code projector

$$P_{\mathrm{SC},L} \;=\; \frac{1}{|S_L|} \sum_{g\in S_L} g. \tag{701}$$

(b) The code encodes a fixed number of logical qubits (typically one), and its distance d_L scales linearly with the linear size L,

$$d_L \;\sim\; L, \tag{702}$$

as in standard planar surface codes [1, 14, 5].

Definition 13.2 (Stabilizer robustness functional)**.** Let $\mathcal{C}_L$ be a surface code as in Definition 13.1, and let $\Lambda : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2)$ be a single–qubit Pauli channel with Kraus form

$$\Lambda(\rho) \;=\; \sum_{P\in\{\mathbb{I},X,Y,Z\}} q_P\, P\rho P, \qquad q_P \geq 0, \quad \sum_P q_P = 1. \tag{703}$$

We define the global noise channel on the register by $\Lambda^{\otimes n_L}$ and the *stabilizer robustness functional* by

$$F_{\mathrm{SC},L}(\Lambda) \;:=\; \Big\langle \psi_{\mathrm{SC},L} \,\Big|\, \Lambda^{\otimes n_L}(P_{\mathrm{SC},L}) \,\Big|\, \psi_{\mathrm{SC},L} \Big\rangle, \tag{704}$$

where $|\psi_{\mathrm{SC},L}\rangle \in \mathcal{C}_L$ is any encoded state. As in the correlation–only analysis, this quantity is independent of the choice of $|\psi_{\mathrm{SC},L}\rangle$ and depends only on the stabilizer weight distribution and the parameters of Λ, in direct analogy with stabilizer–based robustness/fidelity functionals used to characterise code performance under Pauli noise [13, 21, 15, 5].

In the triadic setting, the only change is that Λ is now understood as an effective local Pauli channel on each emergent Twofold fibre, obtained from the triad–split error rates $(p_{\mathrm{corr}}, p_{\mathrm{vac}}, p_{\mathrm{spec}})$ via Lemma 12.3. Concretely, one can either:

- assume approximate spatial homogeneity and take Λ to be a common single–qubit channel for all sites on the patch (e.g. a suitable average of the Λ_j), or
- take Λ to be a worst–case representative (e.g. $\Lambda = \Lambda_j$ for a site x_j with $p^{(j)}_{\mathrm{tot}} = p^{(j)}_{\mathrm{corr}} + p^{(j)}_{\mathrm{vac}} + p^{(j)}_{\mathrm{spec}}$ close to the global maximum),

so that the resulting $F_{\mathrm{SC},L}(\Lambda)$ provides a lower bound on the stabilizer robustness of the code under the full, site–dependent noise on the emergent register. In either case, the dependence of Λ on the triadic Fourfold correlator data of [8] enters only through the effective local error rates $(p_{\mathrm{corr}}, p_{\mathrm{vac}}, p_{\mathrm{spec}})$ and the associated Pauli probabilities $\{q_P\}$, keeping the threshold analysis in the same universality class as standard surface–code thresholds for local Pauli noise.

13.2 Logical error bounds with triadic corrections

We now state a triadic–aware version of the logical error bound for the surface–code family under noise in the Triadic–Fourfold Noise Class, in keeping with the structure of rigorous threshold theorems for local noise on stabilizer codes [1, 2, 3, 16, 4, 5].

Proposition 13.3 (Triadic logical error bound)**.** *Let $\{\mathcal{C}_L\}_{L\geq 1}$ be a surface–code family on the emergent register as in Definition 13.1. Suppose that the physical noise for one QEC cycle on $\mathcal{P}_L$ is described by a noise kernel $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$, and let $(p_{\text{corr}}, p_{\text{vac}}, p_{\text{spec}})$ be the global triad–split error rates defined in Definition 12.2. Let ε_{ps} be the basic leakage parameter of Definition 11.3, and let $\varepsilon_{\text{ps}}^{(\text{corr})}$ be the correlational triad–resolved leakage parameter of Definition 11.4.*

Then there exist positive constants C, c and functions

$$f_{\text{vac}} = f_{\text{vac}}\Big(L;\, p_{\text{vac}}, \xi_{\text{vac}}, t_{\text{vac}}\Big), \qquad f_{\text{spec}} = f_{\text{spec}}\Big(L;\, p_{\text{spec}}, \xi_{\text{spec}}, t_{\text{spec}}\Big), \tag{705}$$

such that, for all sufficiently large L, the logical error probability $p_{\log}^{(L)}$ for one QEC cycle satisfies

$$p_{\log}^{(L)} \;\leq\; C\left(\frac{p_{\text{corr}}}{p_{\text{corr},*}}\right)^{c\,d_L} \;+\; f_{\text{vac}} \;+\; f_{\text{spec}} \;+\; O\Big(\varepsilon_{\text{ps}}^{(\text{corr})}\Big), \tag{706}$$

where:

(a) $p_{\text{corr},*}$ is the usual surface–code threshold for independent local Pauli noise (the critical value of p_{corr} below which the standard threshold theorem holds), as in Refs. [1, 2, 3, 4, 5]. Equivalently, $p_{\text{corr},*}$ is the threshold parameter p_* appearing in the correlation–only IQR threshold theorem (Theorem 7.2).

(b) f_{vac} and f_{spec} satisfy:

  (i) for fixed L, both f_{vac} and f_{spec} are nondecreasing in p_{vac} and p_{spec} respectively, and nonincreasing in the triadic lengths $\xi_{\text{vac}}, \xi_{\text{spec}}$ and times $t_{\text{vac}}, t_{\text{spec}}$, reflecting the fact that longer vacuum/spectral coherence scales reduce inhomogeneity over the patch and over many cycles, in line with slowly varying noise models in open quantum systems [9, 10];

  (ii) in the limit of infinite vacuum and spectral coherence scales,

$$\lim_{\xi_{\text{vac}},\xi_{\text{spec}},t_{\text{vac}},t_{\text{spec}}\to\infty} f_{\text{vac}} = 0, \qquad \lim_{\xi_{\text{vac}},\xi_{\text{spec}},t_{\text{vac}},t_{\text{spec}}\to\infty} f_{\text{spec}} = 0; \tag{707}$$

  (iii) as long as the patch $\mathcal{P}_L$ has diameter much smaller than the relevant vacuum and spectral coherence lengths, and the QEC time window $\Delta t_{\text{QEC}} \ll t_{\text{vac}}, t_{\text{spec}}$ holds (cf. Definition 10.10 and Remark 10.11 and the discussion in [8]), the quantities $f_{\text{vac}}, f_{\text{spec}}$ remain small uniformly in L over the range of interest.

Moreover, the $O\Big(\varepsilon_{\text{ps}}^{(\text{corr})}\Big)$ term in (706) denotes a contribution bounded in absolute value by $C'\,\varepsilon_{\text{ps}}^{(\text{corr})}$ for some constant $C' > 0$ depending only on the code family and on the reconstruction scheme relating $\Delta\hat{G}$ to the one–cycle channel.

Consequently, the Thesis valuation for the logical correctness proposition $\varphi_L^{(L)}$ obeys

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}};\varphi_L^{(L)}) \;=\; 1-p_{\log}^{(L)} \;\geq\; 1-C\Big(\frac{p_{\mathrm{corr}}}{p_{\mathrm{corr},*}}\Big)^{c\,d_L} - f_{\mathrm{vac}} - f_{\mathrm{spec}} - O\Big(\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})}\Big), \tag{708}$$

in harmony with the Tetralemma–based valuation scheme developed in [8].

Proof. The proof proceeds in four steps:

1. reduction of the Triadic–Fourfold noise to a correlation–only IQR noise model with effective local Pauli parameters;
2. identification of the dominant correlational error rate p_{corr} and definition of the correction functions $f_{\mathrm{vac}}, f_{\mathrm{spec}}$;
3. application of the correlation–only IQR threshold theorem;
4. incorporation of leakage and passage to the valuation bound.

Step 1: From Triadic–Fourfold noise to a local Pauli model.

Let $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ be the noise kernel for one QEC cycle on $\mathcal{P}_L$, realised on the fixed QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$ and QEC–realized weave configuration $\mathcal{I}^{[\mathscr{S}]}$.

By Definition 10.5 (Triadic–Fourfold Noise Class) and Lemma 10.6, the Twofold–to–Twofold block $\Delta\hat{G}_{(0,1)\to(0,1)}$ satisfies the spatial and temporal clustering bounds of the correlation–only IQR noise class (Assumption 4.5) for some correlation length $\xi > 0$, correlation time $t_{\mathrm{corr}} > 0$ and smallness parameter $\epsilon_0 > 0$:

$$\Big\|\Delta\hat{G}_{(0,1)\to(0,1)}(x,t;\,x',t')\Big\| \;\leq\; \epsilon_0\,\exp\!\Big(-\frac{d_{\bar{\Sigma}}(x,x')}{\xi} - \frac{|t-t'|}{t_{\mathrm{corr}}}\Big), \tag{709}$$

for all $x,x' \in \mathcal{P}_L$ and t,t' within the QEC cycle. Thus, upon forgetting triadic provenance and the (p,s) sector, the physical noise lies in the correlation–only IQR noise class.

At the level of the emergent register, we now restrict the noise to the Twofold fibres and derive effective single–site channels. For each site $x_j \in \mathcal{P}_L$, let $\mathcal{N}_j^{(1)}$ be the single–site reduced channel on $\mathcal{H}_{(0,1),x_j}$ constructed in Definition 12.1. By Lemma 12.3 (Triadic IQR → effective Pauli noise), there exists a single–qubit Pauli channel

$$\Lambda_j(\rho) \;=\; \sum_{P\in\{\mathbb{I},X,Y,Z\}} q_P^{(j)}\,P\rho P,$$

such that

$$\Big\|\mathcal{N}_j^{(1)} - \Lambda_j\Big\|_{\diamond} \;\leq\; \epsilon_{\mathrm{loc}}^{(j)}, \tag{710}$$

with

$$\epsilon_{\mathrm{loc}}^{(j)} \;\leq\; f_{\mathrm{loc}}\Big(p_{\mathrm{corr}}^{(j)},\,p_{\mathrm{vac}}^{(j)},\,p_{\mathrm{spec}}^{(j)};\,\{\xi_a\},\,\{t_a\}\Big), \tag{711}$$

for some nondecreasing function f_{loc} which depends only on the triadic clustering data and on the IQR–to–channel reconstruction procedure. By definition of the global triad–split error rates,

$$p_{\mathrm{corr}} = \max_j p_{\mathrm{corr}}^{(j)}, \quad p_{\mathrm{vac}} = \max_j p_{\mathrm{vac}}^{(j)}, \quad p_{\mathrm{spec}} = \max_j p_{\mathrm{spec}}^{(j)},$$

we therefore obtain a uniform local Pauli approximation at each site, with diamond–norm errors bounded in terms of $(p_{\mathrm{corr}}, p_{\mathrm{vac}}, p_{\mathrm{spec}})$ and the triadic length and time scales.

Step 2: Dominant correlational error rate and definition of $f_{\mathrm{vac}}, f_{\mathrm{spec}}$.

Let $p^{(j)}_{\mathrm{Pauli}}$ denote the physical error rate of the Pauli channel Λ_j, e.g.

$$p^{(j)}_{\mathrm{Pauli}} := 1 - q^{(j)}_{\mathbb{I}} = q^{(j)}_X + q^{(j)}_Y + q^{(j)}_Z .$$

By Definition 12.2, the deviation of the single–site channel from its ideal decomposes as

$$\mathcal{N}^{(1)}_j - \mathcal{N}^{(1),0}_j = \sum_{a \in \{\mathrm{corr},\mathrm{vac},\mathrm{spec}\}} \Delta\mathcal{N}^{(1,a)}_j ,$$

where each $\Delta\mathcal{N}^{(1,a)}_j$ is reconstructed linearly from the triad–resolved Twofold–preserving block $\Delta\hat{G}^{(a)}_{(0,1)\to(0,1)}$. The triad–split single–site error rates are

$$p^{(j)}_a \;=\; \frac{1}{2} \sup_{\rho_j \in \mathcal{S}(\mathcal{H}_{(0,1),x_j})} \left\| \Delta\mathcal{N}^{(1,a)}_j(\rho_j) \right\|_1, \qquad a \in \{\mathrm{corr}, \mathrm{vac}, \mathrm{spec}\}. \tag{712}$$

Standard relations between trace–norm and diamond–norm deviations of single–qubit channels [7] imply that the Pauli error rate $p^{(j)}_{\mathrm{Pauli}}$ can be bounded in terms of the $p^{(j)}_a$ and $\epsilon^{(j)}_{\mathrm{loc}}$, with constants depending only on the channel reconstruction scheme. In particular, there exist constants $\kappa_{\mathrm{corr}}, \kappa_{\mathrm{vac}}, \kappa_{\mathrm{spec}} > 0$ and a nondecreasing function g such that

$$p^{(j)}_{\mathrm{Pauli}} \;\le\; \kappa_{\mathrm{corr}}\, p^{(j)}_{\mathrm{corr}} + \kappa_{\mathrm{vac}}\, p^{(j)}_{\mathrm{vac}} + \kappa_{\mathrm{spec}}\, p^{(j)}_{\mathrm{spec}} + g\Big(\epsilon^{(j)}_{\mathrm{loc}}\Big). \tag{713}$$

In the regime of interest, the local Pauli approximation errors $\epsilon^{(j)}_{\mathrm{loc}}$ are small whenever $(p^{(j)}_{\mathrm{corr}}, p^{(j)}_{\mathrm{vac}}, p^{(j)}_{\mathrm{spec}})$ are small and the triadic coherence scales are large, so we may absorb $g(\epsilon^{(j)}_{\mathrm{loc}})$ into the contributions proportional to $p^{(j)}_{\mathrm{vac}}$ and $p^{(j)}_{\mathrm{spec}}$ without loss of generality.

Taking the maximum over j and using the global definitions of the error rates, we obtain an effective physical error rate per site

$$p_{\mathrm{eff}} \;:=\; \max_j p^{(j)}_{\mathrm{Pauli}} \;\le\; \kappa_{\mathrm{corr}}\, p_{\mathrm{corr}} + \kappa_{\mathrm{vac}}\, p_{\mathrm{vac}} + \kappa_{\mathrm{spec}}\, p_{\mathrm{spec}}. \tag{714}$$

We now separate out the dominant correlational contribution by writing

$$p_{\mathrm{eff}} \;=\; p_{\mathrm{corr}} + \Delta p_{\mathrm{vac}} + \Delta p_{\mathrm{spec}}, \tag{715}$$

where

$$\Delta p_{\mathrm{vac}} := (\kappa_{\mathrm{corr}} - 1)_+\, p_{\mathrm{corr}} + \kappa_{\mathrm{vac}}\, p_{\mathrm{vac}}, \qquad \Delta p_{\mathrm{spec}} := \kappa_{\mathrm{spec}}\, p_{\mathrm{spec}}, \tag{716}$$

and $(\kappa_{\mathrm{corr}} - 1)_+$ denotes the positive part. By construction, Δp_{vac} and Δp_{spec} are nondecreasing in p_{vac} and p_{spec} respectively, and in the limit where $p_{\mathrm{vac}}, p_{\mathrm{spec}} \to 0$ they vanish.

Moreover, the triadic clustering bounds and the triadic QEC time window (Definitions 10.8 and 10.10) imply that vacuum and spectral contributions vary slowly across the patch and across cycles. In particular, as the vacuum and spectral coherence lengths and times $(\xi_{\mathrm{vac}}, \xi_{\mathrm{spec}}; t_{\mathrm{vac}}, t_{\mathrm{spec}})$ tend to infinity, the corresponding triad–resolved blocks become spatially and temporally homogeneous on the scales relevant for QEC. In this regime, $\mathcal{N}_j^{(1,\mathrm{vac})}$ and $\mathcal{N}_j^{(1,\mathrm{spec})}$ act as quasi–static, almost identical distortions of the ideal channel across the patch, which can be absorbed into a redefinition of the ideal dynamics $\mathcal{N}_j^{(1),0}$ up to corrections that vanish as $\xi_{\mathrm{vac}}, \xi_{\mathrm{spec}}, t_{\mathrm{vac}}, t_{\mathrm{spec}} \to \infty$.

We encode the residual effect of these slowly varying components on the logical error rate via two correction functions f_{vac} and f_{spec}. For fixed

$$\Big(L,\, p_{\mathrm{corr}}, p_{\mathrm{vac}}, p_{\mathrm{spec}}, \xi_{\mathrm{vac}}, \xi_{\mathrm{spec}}, t_{\mathrm{vac}}, t_{\mathrm{spec}}\Big),$$

let $p_{\log}^{(L)}$ be the logical error probability for one cycle under the full Triadic–Fourfold noise, and let $p_{\log,\mathrm{corr}}^{(L)}$ be the logical error probability for the same code family under a purely correlational local Pauli noise model with per–site error rate p_{corr} (and no vacuum/spectral or leakage contributions). We then define

$$f_{\mathrm{vac}} := \sup_{\substack{\text{Triadic–Fourfold noise kernels} \\ \text{with given } L, p_{\mathrm{corr}}, p_{\mathrm{vac}}, \xi_{\mathrm{vac}}, t_{\mathrm{vac}}}} \Big[p_{\log}^{(L)} - p_{\log,\mathrm{corr}}^{(L)}\Big]_+, \tag{717}$$

and analogously f_{spec} for the spectral sector, fixing $(L, p_{\mathrm{corr}}, p_{\mathrm{spec}}, \xi_{\mathrm{spec}}, t_{\mathrm{spec}})$. The dependence of $p_{\log}^{(L)}$ on the microscopic parameters is monotone in the error rates and monotone decreasing in the coherence lengths and times; the same properties are therefore inherited by $f_{\mathrm{vac}}\Big(L; p_{\mathrm{vac}}, \xi_{\mathrm{vac}}, t_{\mathrm{vac}}\Big)$ and $f_{\mathrm{spec}}\Big(L; p_{\mathrm{spec}}, \xi_{\mathrm{spec}}, t_{\mathrm{spec}}\Big)$. The clustering and time–window assumptions ensure that in the limit of infinite vacuum and spectral coherence scales the quasi–static vacuum and spectral distortions can be absorbed entirely into the ideal dynamics, so $f_{\mathrm{vac}}, f_{\mathrm{spec}} \to 0$ in that limit. This establishes the qualitative properties of f_{vac} and f_{spec} listed in the statement of the proposition.

Step 3: Application of the correlation–only IQR threshold theorem.

In the correlation–only setting, Theorem 7.2 states that for the IQR noise class with per–site physical error rate p, process differential parameter δ and leakage parameter $\varepsilon_{\mathrm{ps}}$, there exist constants $C_0, c_0 > 0$ and a threshold value p_* such that, for $p < p_*$ and sufficiently small $\delta, \varepsilon_{\mathrm{ps}}$, the logical error probability under one QEC cycle obeys

$$p_{\log,\mathrm{corr}}^{(L)} \;\leq\; C_0 \left(\frac{p}{p_*}\right)^{c_0 d_L} + O(\varepsilon_{\mathrm{ps}}), \tag{718}$$

for all sufficiently large L, in line with standard surface–code threshold theorems [1, 2, 3, 4, 5]. In our notation, we identify p_* with $p_{\mathrm{corr},*}$.

In the triadic setting, the effective per–site physical error rate entering the correlation–only theorem is p_{eff} given by (714). For fixed $p_{\mathrm{corr}} < p_{\mathrm{corr},*}$ and sufficiently small $p_{\mathrm{vac}}, p_{\mathrm{spec}}$, the quantity p_{eff} is still below the correlation–only threshold $p_{\mathrm{corr},*}$; the corresponding logical error probability is thus bounded by

$$p_{\log}^{(L)} \;\leq\; C_0 \left(\frac{p_{\mathrm{eff}}}{p_{\mathrm{corr},*}}\right)^{c_0 d_L} + O(\varepsilon_{\mathrm{ps}}) + f_{\mathrm{vac}} + f_{\mathrm{spec}}, \tag{719}$$

where we have explicitly separated the vacuum and spectral correction terms into f_{vac} and f_{spec} as discussed in Step 2, and we work in the regime where the process differential parameter (as in Definition 12.6) is sufficiently small so that its contribution can be absorbed into the constants.

Using $p_{\text{eff}} = p_{\text{corr}} + \Delta p_{\text{vac}} + \Delta p_{\text{spec}}$ and the fact that the map $p \mapsto (p/p_{\text{corr},*})^{c_0 d_L}$ is increasing, we may bound

$$\left(\frac{p_{\text{eff}}}{p_{\text{corr},*}}\right)^{c_0 d_L} \;\leq\; \left(\frac{p_{\text{corr}}}{p_{\text{corr},*}}\right)^{c_0 d_L} + \Delta^{(L)}_{\text{vac}} + \Delta^{(L)}_{\text{spec}},$$

for suitable nonnegative quantities $\Delta^{(L)}_{\text{vac}}$ and $\Delta^{(L)}_{\text{spec}}$ which depend monotonically on $(p_{\text{vac}}, \xi_{\text{vac}}, t_{\text{vac}})$ and $(p_{\text{spec}}, \xi_{\text{spec}}, t_{\text{spec}})$ and vanish in the limit of infinite coherence scales. These contributions can be absorbed into redefinitions of f_{vac} and f_{spec} without affecting their qualitative properties. Consequently, there exist constants $C, c > 0$ and functions $f_{\text{vac}}, f_{\text{spec}}$ with the claimed properties such that

$$p^{(L)}_{\log} \;\leq\; C\left(\frac{p_{\text{corr}}}{p_{\text{corr},*}}\right)^{c\,d_L} + f_{\text{vac}} + f_{\text{spec}} + O(\varepsilon_{\text{ps}}), \tag{720}$$

where we have also allowed for a harmless rescaling of the constants and of the threshold $p_{\text{corr},*}$, which are independent of L and of the triadic data.

Step 4: Leakage and passage to the valuation bound.

The correlation–only threshold theorem involves the leakage parameter ε_{ps} defined in Definition 11.3. In the triadic setting we have triad–resolved leakage parameters $\varepsilon^{(a)}_{\text{ps}}$, and Lemma 11.5 shows that

$$\varepsilon_{\text{ps}} \;\leq\; C'\left(\varepsilon^{(\text{corr})}_{\text{ps}} + \varepsilon^{(\text{vac})}_{\text{ps}} + \varepsilon^{(\text{spec})}_{\text{ps}}\right), \tag{721}$$

for some constant $C' \geq 1$. As in the error–rate analysis, the vacuum and spectral leakage contributions are suppressed by the large triadic coherence lengths and times and can be absorbed into f_{vac} and f_{spec}, leaving the correlational component $\varepsilon^{(\text{corr})}_{\text{ps}}$ as the dominant leakage parameter entering the threshold bound. Thus there exists a constant $\tilde{C} > 0$ such that

$$O(\varepsilon_{\text{ps}}) \;=\; O\Big(\varepsilon^{(\text{corr})}_{\text{ps}}\Big) \;\leq\; \tilde{C}\,\varepsilon^{(\text{corr})}_{\text{ps}}, \tag{722}$$

and we obtain

$$p^{(L)}_{\log} \;\leq\; C\left(\frac{p_{\text{corr}}}{p_{\text{corr},*}}\right)^{c\,d_L} + f_{\text{vac}} + f_{\text{spec}} + O\Big(\varepsilon^{(\text{corr})}_{\text{ps}}\Big), \tag{723}$$

which is exactly (706).

Finally, the Thesis valuation for the logical correctness proposition $\varphi^{(L)}_L$ is by definition $v_{\text{Th}}(\rho_{\text{out}}; \varphi^{(L)}_L) = 1 - p^{(L)}_{\log}$, so the previous inequality is equivalent to the valuation bound (708). This completes the proof. □

Remark 13.4 (Triadic correction terms)**.** The bound (706) should be viewed as a triadic–aware refinement of the standard surface–code threshold inequality:

- The first term reproduces the usual exponential suppression of logical errors with distance, provided $p_{\text{corr}} < p_{\text{corr},*}$, as in conventional threshold theorems for local noise on stabilizer codes [1, 2, 3, 4, 5].

- The functions f_{vac} and f_{spec} encode corrections due to finite vacuum and spectral coherence scales and are controlled by the triadic clustering data $(p_{\mathrm{vac}}, \xi_{\mathrm{vac}}, t_{\mathrm{vac}})$ and $(p_{\mathrm{spec}}, \xi_{\mathrm{spec}}, t_{\mathrm{spec}})$ inherited from the QEC–realized weave configuration and Good–clock structure of [8]. In the ideal limit of infinite vacuum and spectral coherence lengths and times, these corrections vanish, and one recovers the purely correlational threshold behaviour.
- The term $O(\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})})$ reflects the impact of fast correlational leakage out of the register subspace, as discussed in Remark 11.6, and parallels the role of leakage/error parameters in standard FTQC analyses with leakage levels [17, 3, 5].

Operationally, the first term can be suppressed by increasing the code distance d_L while keeping $p_{\mathrm{corr}} < p_{\mathrm{corr},*}$, as in the usual surface–code setting. The triadic correction terms $f_{\mathrm{vac}}, f_{\mathrm{spec}}$ and $O(\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})})$ are instead controlled by the ambient triadic data: by working within a sufficiently homogeneous vacuum cell (so that the patch $\mathcal{P}_L$ sees nearly constant vacuum/spectral backgrounds) and enforcing the QEC time window $\Delta t_{\mathrm{QEC}} \ll t_{\mathrm{vac}}, t_{\mathrm{spec}}$, these corrections can be made uniformly small over the range of code sizes of interest. In this regime, the logical performance is dominated by the correlational error rate p_{corr}, and the threshold behaviour closely matches that of conventional surface–code architectures [1, 4, 5], now with triadic provenance information carried along as a controlled perturbation.

13.3 Process holosymmetry with triadic corrections

We now incorporate the triadic decomposition into the process–level (holosymmetry) part of the threshold analysis. This controls the stability of process equivalence between different QEC schedules, in the sense of the dual–history holosymmetry framework developed in the IQR/PQS setting of [8] and in close analogy with channel–overlap techniques used in quantum information theory and FTQC analyses of schedule robustness [6, 2, 3, 5].

Proposition 13.5 (Triadic process holosymmetry bound)**.** *Let A and B be two QEC schedules implementing the same logical QEC cycle on a patch $\mathcal{P}_L$, and let $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ be the noise kernel governing both schedules on the ambient configuration $\mathscr{D}_{\mathrm{QEC}}$ and weave $\mathcal{I}^{[\mathscr{S}]}$. For each site x_j let $\mathbf{q}^{(j,A)}, \mathbf{q}^{(j,B)}$ be the per–site Pauli distributions as in Definition 12.5, and let δ_{tot} be the global process differential parameter defined in (697). Let m_L be the number of register sites touched by the QEC schedule per cycle. The discussion here is the natural triadic refinement of the Pauli–channel holosymmetry theorem in Theorem 6.12 and its dual–history interpretation in the IQR framework [8].*

Then:

(a) *For one cycle, the holosymmetric valuation $v_{\mathrm{Holo}}^{(L)}$ in the dual–history interferometer between schedules A and B satisfies*

$$v_{\mathrm{Holo}}^{(L)} = \frac{1}{2}\Big(1 + \prod_{j=1}^{m_L} k_{\mathrm{tot}}^{(j)}\Big), \tag{724}$$

where $k_{\mathrm{tot}}^{(j)}$ *is the per–site Bhattacharyya overlap defined in* (696)*. In particular, since* $k_{\mathrm{tot}}^{(j)} = 1 - \delta_{\mathrm{tot}}^{(j)}$ *and* $\delta_{\mathrm{tot}} = \max_j \delta_{\mathrm{tot}}^{(j)}$*, we have* $k_{\mathrm{tot}}^{(j)} \geq 1 - \delta_{\mathrm{tot}}$ *for all* j*, and hence*

$$\prod_{j=1}^{m_L} k_{\mathrm{tot}}^{(j)} \;\geq\; (1-\delta_{\mathrm{tot}})^{m_L} \;\geq\; 1 - m_L\delta_{\mathrm{tot}}, \tag{725}$$

where the last inequality is Bernoulli's inequality for $0 \leq \delta_{\mathrm{tot}} \leq 1$*. Substituting into* (724) *yields the bound*

$$1 - v_{\mathrm{Holo}}^{(L)} \;\leq\; \frac{1}{2}\, m_L\, \delta_{\mathrm{tot}}, \tag{726}$$

in direct analogy with the correlation–only holosymmetry bound derived from Pauli overlaps in [8].

(b) For T *successive QEC cycles, under the assumption of stationarity and weakly correlated noise across cycles (so that per–cycle Pauli distributions and overlaps compose multiplicatively up to higher–order corrections, as in standard threshold analyses for local noise models [2, 3, 5]), the multi–cycle holosymmetric valuation* $v_{\mathrm{Holo}}^{(L,T)}$ *satisfies*

$$1 - v_{\mathrm{Holo}}^{(L,T)} \;\lesssim\; \frac{1}{2}\, m_L\, T\, \delta_{\mathrm{tot}}, \tag{727}$$

where $\lesssim$ *indicates that higher–order terms in* $m_L T \delta_{\mathrm{tot}}$ *have been neglected, in the same spirit as the small–parameter expansions used for process distances in open quantum systems [9, 10].*

Moreover, in the triadic setting one has, to leading order,

$$\delta_{\mathrm{tot}} \;\approx\; \delta_{\mathrm{corr}} + \delta_{\mathrm{vac}} + \delta_{\mathrm{spec}}, \tag{728}$$

where $\delta_{\mathrm{corr}}, \delta_{\mathrm{vac}}, \delta_{\mathrm{spec}}$ *are the global triad–resolved differentials of Definition 12.6. The contributions* δ_{vac} *and* δ_{spec} *are typically much smaller than* δ_{corr} *whenever schedules* A *and* B *are implemented on the same QEC diagnostic realized reference configuration and share the same band–limiting and Good–clock structure, as in Assumptions 9.2 and 9.16 and the IQR Good–clock construction of [8].*

Proof. We proceed in three steps: (i) reduction to a product of single–site Pauli overlaps via the Pauli–holosymmetry theorem; (ii) bounding the deviation of $v_{\mathrm{Holo}}^{(L)}$ from 1 in terms of the global differential δ_{tot}; and (iii) extension to T cycles and discussion of the triadic split $\delta_{\mathrm{tot}} \approx \delta_{\mathrm{corr}} + \delta_{\mathrm{vac}} + \delta_{\mathrm{spec}}$.

Step 1: Holosymmetry for one cycle as a product of Bhattacharyya overlaps.

Fix a single QEC cycle of schedules A and B on the patch $\mathcal{P}_L$, and let $x_1, \ldots, x_{m_L}$ be the register sites touched by that cycle. For each site x_j we have the single–site reduced channels $\mathcal{N}_{j,A}^{(1)}$ and $\mathcal{N}_{j,B}^{(1)}$ on the Twofold fibre $\mathcal{H}_{(0,1),x_j}$, constructed as in Definition 12.1 from the common Triadic–Fourfold noise kernel $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$.

By Lemma 12.3 (Triadic IQR → effective Pauli noise), there exist single–qubit Pauli channels $\Lambda_{j,A}, \Lambda_{j,B}$ on $\mathcal{H}_{(0,1),x_j}$ of the form

$$\Lambda_{j,\alpha}(\rho) = \sum_{P\in\{\mathbb{I},X,Y,Z\}} q_P^{(j,\alpha)}\, P\rho P, \qquad \alpha \in \{A, B\},$$

such that

$$\left\|\mathcal{N}^{(1)}_{j,\alpha} - \Lambda_{j,\alpha}\right\|_{\diamond} \text{ is small,}$$

with the smallness controlled by the triad–split error rates and the clustering data. The vectors $\mathbf{q}^{(j,\alpha)} = \{q_P^{(j,\alpha)}\}_P$ are precisely the per–site Pauli distributions of Definition 12.5.

Consider now the dual–history holosymmetry gadget that interferes schedules A and B over one cycle, restricted to the m_L touched register sites (and tracing out ancilla/environment degrees of freedom). In the Pauli picture, this is exactly the set–up of the Pauli–channel holosymmetry theorem (Theorem 6.12): at each site x_j we have a pair of local Pauli channels with error distributions $\mathbf{q}^{(j,A)}, \mathbf{q}^{(j,B)}$, and the dual–history overlap factor for that site is the Bhattacharyya overlap

$$k_{\mathrm{tot}}^{(j)} := \sum_{P \in \{\mathbb{I},X,Y,Z\}} \sqrt{q_P^{(j,A)}\, q_P^{(j,B)}},$$

which is exactly the quantity defined in (696).

By Theorem 6.12, the total dual–history overlap for the one–cycle gadget factorises as the product of these single–site overlaps,

$$C^{(L)} \;=\; \prod_{j=1}^{m_L} k_{\mathrm{tot}}^{(j)}. \tag{729}$$

The corresponding holosymmetric valuation is

$$v_{\mathrm{Holo}}^{(L)} \;=\; \frac{1}{2}\Big(1 + C^{(L)}\Big) \;=\; \frac{1}{2}\Big(1 + \prod_{j=1}^{m_L} k_{\mathrm{tot}}^{(j)}\Big),$$

which is exactly (724).

Step 2: Bounding $1 - v_{\mathrm{Holo}}^{(L)}$ in terms of δ_{tot}.

For each site x_j, Definition 12.6 introduces the per–site total process differential

$$\delta_{\mathrm{tot}}^{(j)} := 1 - k_{\mathrm{tot}}^{(j)},$$

and the global differential parameter

$$\delta_{\mathrm{tot}} := \max_j \delta_{\mathrm{tot}}^{(j)}.$$

By construction,

$$k_{\mathrm{tot}}^{(j)} = 1 - \delta_{\mathrm{tot}}^{(j)} \;\geq\; 1 - \delta_{\mathrm{tot}}, \qquad j = 1, \ldots, m_L. \tag{730}$$

Using (729) and (730) we obtain

$$C^{(L)} = \prod_{j=1}^{m_L} k_{\mathrm{tot}}^{(j)} \;\geq\; \prod_{j=1}^{m_L} (1 - \delta_{\mathrm{tot}}) = (1 - \delta_{\mathrm{tot}})^{m_L}. \tag{731}$$

Substituting into (724) yields the lower bound

$$v_{\mathrm{Holo}}^{(L)} \;\geq\; \frac{1}{2}\Big(1 + (1 - \delta_{\mathrm{tot}})^{m_L}\Big),$$

and hence

$$1 - v_{\mathrm{Holo}}^{(L)} \;\le\; \frac{1}{2}\Big[1 - (1 - \delta_{\mathrm{tot}})^{m_L}\Big]. \tag{732}$$

For $0 \le \delta \le 1$ and $m \in \mathbb{N}$ we have Bernoulli's inequality

$$(1 - \delta)^m \;\ge\; 1 - m\delta,$$

so that

$$1 - (1 - \delta)^m \;\le\; m\delta.$$

Applying this with $\delta = \delta_{\mathrm{tot}}$ and $m = m_L$ to (732) gives

$$1 - v_{\mathrm{Holo}}^{(L)} \;\le\; \frac{1}{2}\, m_L \, \delta_{\mathrm{tot}}, \tag{733}$$

which is the claimed bound (726).

Step 3: Extension to T cycles and triadic split of δ_{tot}.

For T successive QEC cycles, we assume (as stated in the proposition) stationarity and weak inter–cycle correlations: the per–cycle Pauli distributions $\mathbf{q}^{(j,A)}, \mathbf{q}^{(j,B)}$ and overlaps $k_{\mathrm{tot}}^{(j)}$ are approximately the same for each cycle, and higher–order correlations between different cycles can be neglected to leading order. Under this assumption, the Pauli–holosymmetry theorem applied cycle by cycle implies that the total dual–history overlap for T cycles is approximately

$$C^{(L,T)} \;\approx\; \Big(C^{(L)}\Big)^T = \Bigg(\prod_{j=1}^{m_L} k_{\mathrm{tot}}^{(j)}\Bigg)^T.$$

Using (730) we obtain

$$C^{(L,T)} \;\gtrsim\; (1 - \delta_{\mathrm{tot}})^{m_L T}.$$

The corresponding multi–cycle holosymmetric valuation is therefore

$$v_{\mathrm{Holo}}^{(L,T)} \;\approx\; \frac{1}{2}\Big(1 + C^{(L,T)}\Big) \;\gtrsim\; \frac{1}{2}\Big(1 + (1 - \delta_{\mathrm{tot}})^{m_L T}\Big),$$

and hence

$$1 - v_{\mathrm{Holo}}^{(L,T)} \;\lesssim\; \frac{1}{2}\Big[1 - (1 - \delta_{\mathrm{tot}})^{m_L T}\Big]. \tag{734}$$

Applying Bernoulli's inequality again gives

$$1 - v_{\mathrm{Holo}}^{(L,T)} \;\lesssim\; \frac{1}{2}\, m_L T \, \delta_{\mathrm{tot}}, \tag{735}$$

which is the claimed bound (727), with the understanding that we have neglected terms of order $(m_L T \delta_{\mathrm{tot}})^2$ and higher in the small–parameter expansion, in the same spirit as process–distance estimates in open system theory [9, 10].

Finally, we address the triadic split $\delta_{\mathrm{tot}} \approx \delta_{\mathrm{corr}} + \delta_{\mathrm{vac}} + \delta_{\mathrm{spec}}$ and the statement that $\delta_{\mathrm{vac}}, \delta_{\mathrm{spec}} \ll \delta_{\mathrm{corr}}$ under the ambient assumptions.

By Definition 12.6, for each site x_j and schedule $\alpha \in \{A, B\}$ we have triad–resolved Pauli weights $q_P^{(j,\alpha,a)} \ge 0$ such that

$$q_P^{(j,\alpha)} = q_P^{(j,\alpha,\mathrm{corr})} + q_P^{(j,\alpha,\mathrm{vac})} + q_P^{(j,\alpha,\mathrm{spec})},$$

and we define triad–resolved overlaps

$$k_a^{(j)} := \sum_P \sqrt{q_P^{(j,A,a)}\, q_P^{(j,B,a)}}, \qquad \delta_a^{(j)} := 1 - k_a^{(j)},$$

with global quantities $\delta_a := \max_j \delta_a^{(j)}$ for $a \in \{\mathrm{corr}, \mathrm{vac}, \mathrm{spec}\}$.

Remark 12.7 shows that under mild factorisation assumptions on the triad components (essentially, approximate statistical independence between correlational, vacuum, and spectral noise mechanisms), the per–site total overlaps satisfy

$$k_{\mathrm{tot}}^{(j)} \;\approx\; k_{\mathrm{corr}}^{(j)} k_{\mathrm{vac}}^{(j)} k_{\mathrm{spec}}^{(j)},$$

so that, to first order in the small differentials,

$$\delta_{\mathrm{tot}}^{(j)} = 1 - k_{\mathrm{tot}}^{(j)} \;\approx\; \delta_{\mathrm{corr}}^{(j)} + \delta_{\mathrm{vac}}^{(j)} + \delta_{\mathrm{spec}}^{(j)}.$$

Taking maxima over j on both sides and absorbing higher–order corrections into the definitions of the δ_a yields, at leading order,

$$\delta_{\mathrm{tot}} \;\approx\; \delta_{\mathrm{corr}} + \delta_{\mathrm{vac}} + \delta_{\mathrm{spec}},$$

which is the content of (728).

Because schedules A and B are implemented on the same QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$ and the same QEC–realized weave $\mathcal{I}^{[\mathscr{S}]}$ (Assumptions 9.2 and 9.16), they share the same Good–clock $C = (\hat{T}, U_t)$, the same plateau band–limiting maps, and the same triad projectors. The vacuum and spectral sectors are therefore quasi–static and nearly identical for A and B on the QEC time scales (Definition 10.10 and Remark 10.11), so the corresponding differences in Pauli weights $q_P^{(j,A,\mathrm{vac})}$ vs. $q_P^{(j,B,\mathrm{vac})}$ and similarly for the spectral sector are very small, implying $\delta_{\mathrm{vac}}, \delta_{\mathrm{spec}} \ll \delta_{\mathrm{corr}}$. By contrast, the correlational sector is sensitive to the detailed gate and measurement structure of the schedules and therefore carries the dominant contribution to the process differential, in line with the discussion in Remark 12.4 and with standard treatments of schedule robustness in FTQC [2, 3, 5].

This completes the proof of the triadic process holosymmetry bounds (726) and (727) and of the triadic decomposition (728). □

Corollary 13.6 (Correlation–only limit of triadic process holosymmetry)**.** *In addition to the hypotheses of Proposition 13.5, suppose that the noise kernel $\Delta\hat{G}$ lies in the correlation–only IQR noise class in the sense of Assumption 4.5, i.e. that the vacuum and spectral triad contributions are negligible so that, in the notation of Definition 12.6,*

$$q_P^{(j,\alpha,\mathrm{vac})} \;\approx\; 0, \qquad q_P^{(j,\alpha,\mathrm{spec})} \;\approx\; 0, \tag{736}$$

for all $P \in \{\mathbb{I}, X, Y, Z\}$, all sites x_j and both schedules $\alpha \in \{A, B\}$. Then:

(a) The triad–resolved differentials satisfy

$$\delta_{\mathrm{vac}} \;\approx\; 0, \qquad \delta_{\mathrm{spec}} \;\approx\; 0, \tag{737}$$

so that, to leading order in the small differentials,

$$\delta_{\mathrm{tot}} \;\approx\; \delta_{\mathrm{corr}}. \tag{738}$$

(b) The single–cycle holosymmetry bound (726) reduces to

$$1 - v_{\mathrm{Holo}}^{(L)} \;\leq\; \frac{1}{2}\, m_L\, \delta_{\mathrm{corr}}, \tag{739}$$

which is exactly the Pauli–channel holosymmetry bound for the correlation–only setting as stated in Theorem 6.12 and its IQR dual–history reinterpretation in [8].

(c) Likewise, the multi–cycle bound (727) becomes

$$1 - v_{\mathrm{Holo}}^{(L,T)} \;\lesssim\; \frac{1}{2}\, m_L\, T\, \delta_{\mathrm{corr}}, \tag{740}$$

up to higher–order corrections in the small parameter $m_L T \delta_{\mathrm{corr}}$, again in agreement with the correlation–only holosymmetry estimates in the IQR/PQS framework.

In particular, in the correlation–only regime the triadic process holosymmetry bound of Proposition 13.5 reproduces the original Pauli–channel holosymmetry theorem used in Part A of the analysis.

Remark 13.7 (Consistency with Part A holosymmetry)**.** Corollary 13.6 makes explicit that the triadic formalism does not alter the underlying holosymmetry mechanism: in the regime where vacuum and spectral channels are negligible (or can be fully absorbed into the ideal dynamics), the only surviving contribution to the process differential is the correlational one, $\delta_{\mathrm{tot}} \approx \delta_{\mathrm{corr}}$, and the dual–history overlap is governed solely by the per–site Pauli overlaps associated with the correlation–only IQR noise class. Thus the triadic analysis is a genuine refinement of the Part A holosymmetry picture rather than a competing criterion: it identifies which parts of the microscopic PQS/Triadic–Fourfold structure contribute to the Pauli overlaps, while reducing exactly to the earlier correlation–only holosymmetry bound when triadic provenance is coarse–grained away.

Definition 13.8 (Triadic process threshold condition)**.** Let $\epsilon > 0$, and let m_L and T be as in Proposition 13.5. We say that the noise model satisfies the *triadic process threshold condition* at accuracy ϵ if

$$\delta_{\mathrm{tot}} \;\leq\; \frac{2\epsilon}{m_L T}, \tag{741}$$

where δ_{tot} is related to the triad–resolved differentials by the approximate splitting (728). Under this condition, the multi–cycle holosymmetry bound (727) implies

$$1 - v_{\mathrm{Holo}}^{(L,T)} \;\lesssim\; \epsilon, \tag{742}$$

up to higher–order corrections in the small parameter $m_L T \delta_{\mathrm{tot}}$. In particular, the process–equivalence proposition $\varphi_{\mathrm{proc}}^{(L)}$ remains in the Thesis corner of the Tetralemma with error at most ϵ over T cycles, in the sense of the valuation framework introduced in [8].

Remark 13.9 (Triadic control of process differential)**.** Because schedules A and B share the same QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$, Good–clock, and band–limiting structure, the vacuum and spectral contributions δ_{vac} and δ_{spec} are typically small and can be bounded directly in terms of the triadic spatial and temporal scales associated with those sectors, namely the distances d_a and clustering lengths ξ_a and times t_a for $a \in \{\text{vac}, \text{spec}\}$, together with compiler–level choices ensuring that both schedules experience essentially the same vacuum and spectral backgrounds. This is closely analogous to the way slowly varying background fields and adiabatic drifts are treated in open quantum systems and slow–noise models [9, 10], and to compiler strategies that align schedules to common control and calibration backgrounds in FTQC architectures [2, 3, 5].

In this sense, the process threshold condition (741) is primarily a constraint on the correlational component δ_{corr}, with vacuum and spectral contributions providing subleading corrections controlled by the triadic geometry and Good–clock scales of the QEC–realized weave configuration [8]. The approximate splitting (728) then makes explicit how small δ_{vac} and δ_{spec} must be, relative to δ_{corr}, to keep δ_{tot} below the process threshold.

14 Triadic–Fourfold FTQC Threshold Theorem

We now combine the structures and bounds developed in Sections 9–13 into a single triadic–aware fault–tolerance threshold theorem for surface–code quantum computation in the IQR framework, in the spirit of standard threshold results for FTQC [2, 3, 5] and the correlation–only IQR analysis of [8].

Theorem 14.1 (Triadic–Fourfold IQR fault–tolerance threshold for QC)**.** *Let $\mathscr{D}_{\text{QEC}}$ be a QEC diagnostic realized reference configuration as in Definition 9.1, and let $\mathcal{I}^{[\mathscr{S}]}$ be a QEC–realized weave configuration as in Definition 9.14, with Good–clock $C = (\hat{T}, U_t)$ and triad projectors $\Pi^{(\text{spec})}, \Pi^{(\text{vac})}, \Pi^{(\text{corr})}$. Let $\{\mathcal{C}_L\}_{L \geq 1}$ be a surface–code family on the emergent register as in Definition 13.1, in line with standard surface–code constructions for FTQC [1, 4, 5].*

Assume that for each code size L and each QEC schedule under consideration, the physical noise for one QEC cycle on the patch $\mathcal{P}_L$ is generated by a full fourfold inter–modal correlator $\hat{G}_\Lambda$ with ideal counterpart $\hat{G}^0_\Lambda$, such that the noise kernel $\Delta\hat{G} := \hat{G}_\Lambda - \hat{G}^0_\Lambda$ belongs to the Triadic–Fourfold Noise Class $\mathcal{N}_{\text{tri-4F}}$ (Definition 10.5), extending the correlation–only IQR noise class of [8].

Then there exist positive constants

$$p_{\text{corr},*}, \quad p_{\text{vac},*}, \quad p_{\text{spec},*}, \quad \varepsilon_{\text{ps},*}, \quad c, \; C, \tag{743}$$

such that the following holds.

Let $(p_{\text{corr}}, p_{\text{vac}}, p_{\text{spec}})$ be the effective triad–split error rates associated with $\Delta\hat{G}$ as in Definition 12.2, let $\varepsilon^{(a)}_{\text{ps}}$ be the triad–resolved leakage parameters of Definition 11.4, and let $\delta_{\text{corr}}, \delta_{\text{vac}}, \delta_{\text{spec}}$ be the global process differentials of Definition 12.6, with

$$\delta_{\text{tot}} \; := \; \delta_{\text{corr}} + \delta_{\text{vac}} + \delta_{\text{spec}}. \tag{744}$$

Suppose that, for all sufficiently large L, the following conditions hold:

(A1) Triadic clustering and QEC time window.

The triadic clustering bounds of Assumption 10.9 and the triadic QEC time–window inequality of Definition 10.10 hold, i.e.

$$t_{\text{corr}} \; \ll \; \Delta t_{\text{QEC}} \; \ll \; t_{\text{vac}}, \; t_{\text{spec}}, \tag{745}$$

so that the effective dynamics is quasi–Markovian on QEC time scales in the correlational channel, while vacuum and spectral backgrounds vary slowly in the sense of [9, 10, 8].

(A2) Small triad–split error and leakage parameters.
The effective Twofold–preserving noise parameters satisfy

$$p_{\mathrm{corr}} < p_{\mathrm{corr},*}, \qquad p_{\mathrm{vac}} < p_{\mathrm{vac},*}, \qquad p_{\mathrm{spec}} < p_{\mathrm{spec},*}, \tag{746}$$

and the triad–resolved leakages obey

$$\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})} + \varepsilon_{\mathrm{ps}}^{(\mathrm{vac})} + \varepsilon_{\mathrm{ps}}^{(\mathrm{spec})} \ < \ \varepsilon_{\mathrm{ps},*}, \tag{747}$$

in direct analogy with the small–parameter regimes assumed in conventional threshold theorems for local noise and leakage [17, 2, 3, 5]. Here $p_{\mathrm{corr},*}$ is the usual surface–code threshold for independent local Pauli noise, while $p_{\mathrm{vac},*}, p_{\mathrm{spec},*}, \varepsilon_{\mathrm{ps},*}$ bound the size of the triadic correction terms.

(A3) Small triadic process differential.
For any pair of QEC schedules A, B implementing the same logical QEC cycle on $\mathcal{P}_L$, and for any logical circuit of length $T \leq \mathrm{poly}(d_L)$, the global process differential satisfies the triadic process threshold condition of Definition 13.8, namely

$$\delta_{\mathrm{tot}} \ \leq \ \frac{2\epsilon}{m_L T}, \tag{748}$$

for a prescribed accuracy $\epsilon > 0$, where m_L denotes the number of register sites touched by the schedule per cycle, in the spirit of process–level robustness conditions in holonomic or interferometric formulations of FTQC [2, 5, 8].

Then, for every $\epsilon > 0$, there exists a code distance d_L (equivalently, a code size L) such that:

(C1) Logical robustness.
For one QEC cycle, the logical error probability $p_{\log}^{(L)}$ satisfies

$$p_{\log}^{(L)} \ \leq \ C \left(\frac{p_{\mathrm{corr}}}{p_{\mathrm{corr},*}} \right)^{c\, d_L} \ + \ f_{\mathrm{vac}} \ + \ f_{\mathrm{spec}} \ + \ O\Big(\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})}\Big) \ \leq \ \epsilon, \tag{749}$$

where $f_{\mathrm{vac}}, f_{\mathrm{spec}}$ are the triadic correction terms of Proposition 13.3, controlled by $(p_{\mathrm{vac}}, p_{\mathrm{spec}}, \xi_{\mathrm{vac}}, \xi_{\mathrm{spec}}, t_{\mathrm{vac}}, t_{\mathrm{spec}})$. Equivalently, the Thesis valuation for the logical correctness proposition $\varphi_L^{(L)}$ obeys

$$v_{\mathrm{Th}}\big(\rho_{\mathrm{out}}; \varphi_L^{(L)}\big) \ = \ 1 - p_{\log}^{(L)} \ \geq \ 1 - \epsilon. \tag{750}$$

This reproduces the usual exponential suppression of logical error with distance for $p_{\mathrm{corr}} < p_{\mathrm{corr},*}$, as in standard surface–code and FTQC threshold analyses [1, 2, 4, 5], now augmented by the triadic correction terms.

(C2) Process robustness (holosymmetry).

For any two schedules A, B implementing the same logical circuit over T cycles with $T \leq \mathrm{poly}(d_L)$, the holosymmetric valuation in the dual–history interferometer satisfies

$$v_{\mathrm{Holo}}^{(L,T)} \geq 1 - \epsilon, \tag{751}$$

in accordance with the triadic process holosymmetry bounds of Proposition 13.5 and the process–threshold condition of Definition 13.8.

In particular, for noise in the Triadic–Fourfold Noise Class with parameters below the critical values $(p_{\mathrm{corr},*}, p_{\mathrm{vac},*}, p_{\mathrm{spec},*}, \varepsilon_{\mathrm{ps},*})$ and process differential below the triadic process threshold (748), both the logical proposition *$\varphi_L^{(L)}$ and the* process–equivalence proposition *$\varphi_{\mathrm{proc}}^{(L)}$ remain in the Thesis corner of the Tetralemma with error at most ϵ. The corresponding threshold conditions are determined by the triadic distances and times and by the full fourfold inter–modal correlator, extending the correlation–only IQR threshold of [8] to the triadic–Fourfold setting.*

Proof. The proof combines the constructions and estimates developed in Sections 9–13. We split it into two main parts: logical robustness and process robustness, and then combine them in the Tetralemma valuation language.

Step 1: From Triadic–Fourfold data to local Pauli noise.

By hypothesis, for each code size L and each QEC schedule, the physical noise for one cycle on $\mathcal{P}_L$ is described by a fourfold inter–modal correlator $\hat{G}_\Lambda$ with ideal counterpart $\hat{G}_\Lambda^0$, such that the noise kernel $\Delta\hat{G} := \hat{G}_\Lambda - \hat{G}_\Lambda^0$ belongs to the Triadic–Fourfold Noise Class $\mathcal{N}_{\text{tri-4F}}$ (Definition 10.5).

(1a) Reduction to the correlation–only IQR class. By Lemma 10.6, the triad–resolved clustering and smallness conditions in Definition 10.5 imply that the total Twofold–preserving block $\Delta\hat{G}_{(0,1)\to(0,1)}$ satisfies the correlation–only IQR noise bound of Assumption 4.5 with parameters $(\xi, t_{\mathrm{corr}}, \epsilon_0)$ constructed from the triadic data. Thus, at the level of the Twofold sector, the present Triadic–Fourfold class is a refinement of the correlation–only IQR noise class used in the earlier Phases 1–3 analysis and in [8], and all correlation–only constructions (emergent register, IQR→Pauli reduction, etc.) are available.

(1b) Emergent register and single–site reduced channels. Using the QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$ and the QEC–realized weave $\mathcal{I}^{[\mathscr{S}]}$, we construct, for each patch $\mathcal{P}_L \subset \mathfrak{B}$, the emergent Twofold register $\mathcal{H}_{\mathrm{reg}}^{(L)}$ as in Definition 11.1, together with the register and complement projectors $P_{\mathrm{reg}}^{(L)}, P_{\mathrm{comp}}^{(L)}$ (Definition 11.2). The full noisy QEC cycle defines a CPTP map

$$\mathcal{N} : \mathcal{B}(\mathcal{H}_{\mathrm{phys}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}}),$$

which already incorporates any ancilla and environment degrees of freedom relevant at the cycle level. For each site $x_j \in \mathcal{P}_L$, we obtain the single–site reduced channel

$$\mathcal{N}_j^{(1)} : \mathcal{S}(\mathcal{H}_{(0,1),x_j}) \to \mathcal{S}(\mathcal{H}_{(0,1),x_j})$$

via the embedding/tracing procedure of Definition 12.1, using a fixed reference state ω_{rest} on the remaining Twofold fibres and all nonregister degrees of freedom.

The basic leakage parameter ε_{ps} is defined by the trace–norm deviation between the reduced register dynamics induced by $\mathcal{N}$ and the confined register channel $\mathcal{N}_{\text{reg}}$ (Definition 11.3), and the triad–resolved leakage parameters $\varepsilon_{\text{ps}}^{(a)}$ are introduced in Definition 11.4, with their relation to ε_{ps} quantified in Lemma 11.5.

(1c) Triadic IQR → local Pauli channels. By Lemma 12.3, applied to $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$, the triadic clustering assumptions (Assumption 10.9) and the triadic QEC time window (Definition 10.10) ensure that for each site x_j there exists a single–qubit Pauli channel

$$\Lambda_j : \mathcal{S}(\mathcal{H}_{(0,1),x_j}) \to \mathcal{S}(\mathcal{H}_{(0,1),x_j})$$

of the form

$$\Lambda_j(\rho) \;=\; \sum_{P\in\{\mathbb{I},X,Y,Z\}} q_P^{(j)}\, P\rho P, \qquad q_P^{(j)} \geq 0, \quad \sum_P q_P^{(j)} = 1,$$

such that the diamond–norm distance between the true reduced channel and its Pauli approximation is bounded by

$$\left\|\mathcal{N}_j^{(1)} - \Lambda_j\right\|_\diamond \;\leq\; \epsilon_{\text{loc}}^{(j)} \;\leq\; f_{\text{loc}}\Big(p_{\text{corr}}^{(j)}, p_{\text{vac}}^{(j)}, p_{\text{spec}}^{(j)}; \{\xi_a\}, \{t_a\}\Big),$$

where the triad–split single–site error rates $p_a^{(j)}$ are defined in Definition 12.2. The global error rates $(p_{\text{corr}}, p_{\text{vac}}, p_{\text{spec}})$ are obtained by maximising over sites in $\mathcal{P}_L$, as in Definition 12.2.

Thus, the physical noise for one cycle on $\mathcal{P}_L$ is well approximated, at the single–site level, by a local Pauli noise model with per–site Pauli distributions $\mathbf{q}^{(j)} = \{q_P^{(j)}\}_P$, together with small, triad–controlled corrections quantified by f_{loc}.

Step 2: Logical threshold from surface–code robustness.

We now use the effective local Pauli description to control the logical error probability for one cycle.

(2a) Application of the triadic logical error bound. Proposition 13.3 applies to the surface–code family $\{\mathcal{C}_L\}$ and the triad–split error rates $(p_{\text{corr}}, p_{\text{vac}}, p_{\text{spec}})$ under the Triadic–Fourfold noise assumptions. It states that there exist constants $C, c > 0$ and functions

$$f_{\text{vac}} = f_{\text{vac}}\Big(L; p_{\text{vac}}, \xi_{\text{vac}}, t_{\text{vac}}\Big), \qquad f_{\text{spec}} = f_{\text{spec}}\Big(L; p_{\text{spec}}, \xi_{\text{spec}}, t_{\text{spec}}\Big),$$

such that the logical error probability $p_{\text{log}}^{(L)}$ for one QEC cycle obeys

$$p_{\text{log}}^{(L)} \;\leq\; C\left(\frac{p_{\text{corr}}}{p_{\text{corr},*}}\right)^{c\,d_L} \;+\; f_{\text{vac}} \;+\; f_{\text{spec}} \;+\; O\Big(\varepsilon_{\text{ps}}^{(\text{corr})}\Big), \tag{752}$$

where $p_{\text{corr},*}$ is the usual surface–code threshold for independent local Pauli noise, and the correction terms f_{vac}, f_{spec}, and $O(\varepsilon_{\text{ps}}^{(\text{corr})})$ are controlled by the vacuum and spectral triadic scales and by the leakage parameters (cf. Remark 13.4).

(2b) Using the smallness assumptions on triad–split errors and leakage. By hypothesis (A2) of the theorem, the effective Twofold–preserving noise parameters satisfy

$$p_{\text{corr}} < p_{\text{corr},*}, \qquad p_{\text{vac}} < p_{\text{vac},*}, \qquad p_{\text{spec}} < p_{\text{spec},*},$$

for fixed critical values $p_{\mathrm{vac},*}, p_{\mathrm{spec},*}$, and the triad–resolved leakages obey

$$\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})} + \varepsilon_{\mathrm{ps}}^{(\mathrm{vac})} + \varepsilon_{\mathrm{ps}}^{(\mathrm{spec})} \;<\; \varepsilon_{\mathrm{ps},*},$$

with $\varepsilon_{\mathrm{ps},*} > 0$. Using the monotonicity properties of f_{vac} and f_{spec} with respect to $p_{\mathrm{vac}}, p_{\mathrm{spec}}$ and the triadic lengths and times (as spelled out in Proposition 13.3), and the standard treatment of leakage as a small parameter in threshold analyses [17, 3, 5], we may choose $p_{\mathrm{vac},*}, p_{\mathrm{spec},*}, \varepsilon_{\mathrm{ps},*}$ so that, for a given target accuracy $\epsilon > 0$,

$$f_{\mathrm{vac}}\Big(L; p_{\mathrm{vac}}, \xi_{\mathrm{vac}}, t_{\mathrm{vac}}\Big) \;\le\; \frac{\epsilon}{3}, \qquad f_{\mathrm{spec}}\Big(L; p_{\mathrm{spec}}, \xi_{\mathrm{spec}}, t_{\mathrm{spec}}\Big) \;\le\; \frac{\epsilon}{3}, \qquad O\Big(\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})}\Big) \;\le\; \frac{\epsilon}{3} \tag{753}$$

for all sufficiently large L.

(2c) Choosing the code distance. The remaining term in (752) is

$$C\left(\frac{p_{\mathrm{corr}}}{p_{\mathrm{corr},*}}\right)^{c\,d_L}.$$

Since $p_{\mathrm{corr}}/p_{\mathrm{corr},*} < 1$ is fixed for the given noise model, the map

$$d_L \;\mapsto\; C\left(\frac{p_{\mathrm{corr}}}{p_{\mathrm{corr},*}}\right)^{c\,d_L}$$

decays exponentially in d_L. Hence there exists a distance d_0 such that for all $d_L \ge d_0$,

$$C\left(\frac{p_{\mathrm{corr}}}{p_{\mathrm{corr},*}}\right)^{c\,d_L} \;\le\; \frac{\epsilon}{3}. \tag{754}$$

Equivalently, choosing the code size L large enough so that $d_L \ge d_0$ achieves this bound.

Combining (753) and (754) with (752) yields

$$p_{\log}^{(L)} \;\le\; \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \;=\; \epsilon.$$

Thus for every $\epsilon > 0$, provided the triad–split noise parameters lie below the critical values $(p_{\mathrm{corr},*}, p_{\mathrm{vac},*}, p_{\mathrm{spec},*}, \varepsilon_{\mathrm{ps},*})$, there exists a code distance d_L (equivalently, code size L) such that the logical error probability satisfies $p_{\log}^{(L)} \le \epsilon$.

Rewriting this in Tetralemma valuation form gives

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_L^{(L)}) \;=\; 1 - p_{\log}^{(L)} \;\ge\; 1 - \epsilon,$$

which is precisely the logical robustness statement (C1) of the theorem.

Step 3: Process robustness via triadic process holosymmetry.

We now prove the holosymmetry (process–equivalence) part of the theorem.

(3a) Triad–resolved process differentials. For any pair of QEC schedules A, B implementing the same logical QEC cycle on $\mathcal{P}_L$, the Pauli approximations $\{\Lambda_{j,A}\}_j, \{\Lambda_{j,B}\}_j$ of their single–site reduced channels lead to per–site Pauli distributions $\mathbf{q}^{(j,A)}, \mathbf{q}^{(j,B)}$ as in Definition 12.5. Using the triadic structure of the noise kernel, these distributions admit triad–resolved

contributions $\mathbf{q}^{(j,\alpha,a)}$ for $\alpha \in \{A, B\}$, $a \in \{\text{corr}, \text{vac}, \text{spec}\}$, as in Definition 12.6. The associated Bhattacharyya overlaps and process differentials $\delta_a^{(j)}$ are defined in (694)–(695), and the global quantities $\delta_a, \delta_{\text{tot}}$ are obtained by maximising over sites:

$$\delta_a := \max_j \delta_a^{(j)}, \qquad \delta_{\text{tot}} := \max_j \delta_{\text{tot}}^{(j)} \;=\; \max_j \Big(1 - k_{\text{tot}}^{(j)}\Big),$$

with $k_{\text{tot}}^{(j)}$ as in (696). In the regime where the triad contributions behave approximately independently, Remark 12.7 shows that $\delta_{\text{tot}} \approx \delta_{\text{corr}} + \delta_{\text{vac}} + \delta_{\text{spec}}$, justifying the decomposition used in the theorem.

(3b) Single–cycle holosymmetry bound. Proposition 13.5(a) states that, for one cycle, the holosymmetric valuation $v_{\text{Holo}}^{(L)}$ in the dual–history interferometer for schedules A, B obeys

$$v_{\text{Holo}}^{(L)} \;=\; \frac{1}{2}\Big(1 + (k_{\text{tot}})^{m_L}\Big),$$

where m_L is the number of register sites touched by the schedule per cycle, and k_{tot} is the global Bhattacharyya overlap associated with the per–site distributions. Writing $k_{\text{tot}} = 1 - \delta_{\text{tot}}$ and using the elementary inequality $(1 - \delta_{\text{tot}})^{m_L} \geq 1 - m_L\delta_{\text{tot}}$ for $0 \leq \delta_{\text{tot}} \leq 1$, we obtain

$$v_{\text{Holo}}^{(L)} \;\geq\; \frac{1}{2}\Big(1 + 1 - m_L\delta_{\text{tot}}\Big) \;=\; 1 - \frac{1}{2} m_L \delta_{\text{tot}},$$

or equivalently

$$1 - v_{\text{Holo}}^{(L)} \;\leq\; \frac{1}{2} m_L \delta_{\text{tot}}.$$

(3c) Multi–cycle holosymmetry bound. For T successive QEC cycles implementing the same logical circuit with schedules A, B, Proposition 13.5(b) asserts that, under stationarity and weak inter–cycle correlations, the multi–cycle holosymmetric valuation $v_{\text{Holo}}^{(L,T)}$ satisfies

$$1 - v_{\text{Holo}}^{(L,T)} \;\lesssim\; \frac{1}{2}\, m_L\, T\, \delta_{\text{tot}},$$

where $\lesssim$ indicates that higher–order terms in $m_L T \delta_{\text{tot}}$ have been neglected (the usual small–noise expansion for process distances in open system dynamics [9, 10]). For sufficiently small δ_{tot}, these higher–order contributions can be absorbed into a harmless redefinition of the overall constant or of ϵ.

(3d) Imposing the triadic process threshold. By hypothesis (A3) of the theorem, for any pair of QEC schedules and for any logical circuit of length $T \leq \text{poly}(d_L)$, the global process differential satisfies

$$\delta_{\text{tot}} \;\leq\; \frac{2\epsilon}{m_L T}.$$

This is exactly the triadic process threshold condition of Definition 13.8, with accuracy $\epsilon > 0$. Substituting this into the linear bound gives

$$1 - v_{\text{Holo}}^{(L,T)} \;\lesssim\; \frac{1}{2} m_L\, T \cdot \frac{2\epsilon}{m_L T} \;=\; \epsilon,$$

up to the higher–order corrections, which are negligible in the small–noise regime and can again be absorbed into the choice of ϵ.

Thus, for all $T \le \mathrm{poly}(d_L)$,

$$v_{\mathrm{Holo}}^{(L,T)} \;\ge\; 1-\epsilon,$$

which is the process robustness (holosymmetry) statement (C2) of the theorem.

Step 4: Combined Tetralemma valuations and conclusion.

We now combine Steps 2 and 3.

Logical correctness. From Step 2, for every $\epsilon > 0$ and for any noise model in the Triadic–Fourfold Noise Class with triad–split error and leakage parameters below the critical values $(p_{\mathrm{corr},*}, p_{\mathrm{vac},*}, p_{\mathrm{spec},*}, \varepsilon_{\mathrm{ps},*})$, there exists a code distance d_L such that $p_{\log}^{(L)} \le \epsilon$. Hence the Thesis valuation for the logical correctness proposition $\varphi_L^{(L)}$ obeys

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}};\varphi_L^{(L)}) \;=\; 1 - p_{\log}^{(L)} \;\ge\; 1-\epsilon.$$

Process equivalence. From Step 3, imposing the triadic process threshold condition $\delta_{\mathrm{tot}} \le 2\epsilon/(m_L T)$ ensures that for all $T \le \mathrm{poly}(d_L)$ the holosymmetric valuation satisfies

$$v_{\mathrm{Holo}}^{(L,T)} \;\ge\; 1-\epsilon,$$

so the process–equivalence proposition $\varphi_{\mathrm{proc}}^{(L)}$ also lies in the Thesis corner with error at most ϵ.

Thus, under the stated assumptions on triadic clustering, triadic QEC time window, small triad–split error and leakage parameters, and small triadic process differential, there exists a code distance d_L (equivalently, a code size L) such that both

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}};\varphi_L^{(L)}) \ge 1-\epsilon, \qquad v_{\mathrm{Holo}}^{(L,T)} \ge 1-\epsilon$$

for all logical circuits of length $T \le \mathrm{poly}(d_L)$. This is exactly the Triadic–Fourfold IQR fault–tolerance threshold statement: the logical correctness and process–equivalence propositions remain in the Thesis corner of the Tetralemma with error at most ϵ, with threshold conditions determined by the triadic distances and times and by the full fourfold inter–modal correlator, thereby extending the correlation–only IQR threshold of [8] to the full Triadic–Fourfold setting. □

Remark 14.2 (Dependency structure of the Triadic–Fourfold threshold proof)**.** For the reader's convenience, we summarise the logical structure of Theorem 14.1 and its dependencies:

- *Triadic Fourfold noise ⇒ correlation–only IQR class.* The starting point is the assumption $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ (Definition 10.5), together with triadic clustering and the triadic QEC time window (Assumption 10.9, Definition 10.10). Lemma 10.6 then shows that the Twofold–preserving block $\Delta\hat{G}_{(0,1)\to(0,1)}$ lies in the correlation–only IQR noise class (Assumption 4.5), so all correlation–only constructions from [8] apply.

- *Emergent register, leakage, and register restriction.* The QEC diagnostic realized reference configuration and realized weave (Definitions 9.1 and 9.14, together with Assumptions 9.2 and 9.16) yield the emergent register $\mathcal{H}^{(L)}_{\mathrm{reg}}$ (Definition 11.1) and the projectors $P^{(L)}_{\mathrm{reg}}, P^{(L)}_{\mathrm{comp}}$ (Definition 11.2). These enter the definition of the basic leakage norm (Definition 11.3) and its triad–resolved refinement (Definition 11.4), whose relation is quantified by Lemma 11.5.

- *Single–site channels, triad–split error rates, and local Pauli reduction.* The single–site reduced channels $\mathcal{N}^{(1)}_j$ are defined from the global cycle map $\mathcal{N}$ via Definition 12.1. The Triadic–Fourfold structure then induces the triad–split single–site error rates $p^{(j)}_a$ and global rates $(p_{\mathrm{corr}}, p_{\mathrm{vac}}, p_{\mathrm{spec}})$ (Definition 12.2). Lemma 12.3, together with approximate Pauli covariance and the IQR–to–channel reconstruction, provides effective single–qubit Pauli channels Λ_j approximating $\mathcal{N}^{(1)}_j$, with errors controlled by the $p^{(j)}_a$ and the triadic scales (ξ_a, t_a).

- *Logical error bound from the surface–code family.* The surface–code family on the emergent register is specified in Definition 13.1, and its robustness under local Pauli noise is encoded in the stabilizer robustness functional of Definition 13.2. Proposition 13.3 combines:

  - the local Pauli approximation (Lemma 12.3 and Definition 12.2),
  - the correlation–only IQR threshold theorem (Theorem 7.2),
  - and leakage control via Lemma 11.5,

  to yield the bound $p^{(L)}_{\log}$ in terms of $p_{\mathrm{corr}}, p_{\mathrm{vac}}, p_{\mathrm{spec}}$ and $\varepsilon^{(\mathrm{corr})}_{\mathrm{ps}}$, as used in Step 2 of the proof of Theorem 14.1.

- *Process holosymmetry and triadic process differentials.* At the process level, the Pauli approximations for two schedules A, B yield per–site Pauli distributions $\mathbf{q}^{(j,A)}, \mathbf{q}^{(j,B)}$ (Definition 12.5). The triad–resolved Pauli weights and overlaps are defined in Definition 12.6, with their physical interpretation discussed in Remark 12.7. The Pauli–channel holosymmetry theorem (Theorem 6.12), applied to these local Pauli channels, underlies Proposition 13.5, which expresses the holosymmetric valuation in terms of the per–site Bhattacharyya overlaps. The triadic process threshold condition (Definition 13.8) then feeds directly into Step 3 of the proof of Theorem 14.1.

- *Tetralemma valuations and the final threshold statement.* Finally, the logical and process bounds from Proposition 13.3 and Proposition 13.5, combined with the process threshold condition (Definition 13.8), are rephrased in the Tetralemma valuation framework of [8] to give the inequalities $v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi^{(L)}_L) \geq 1 - \epsilon$ and $v^{(L,T)}_{\mathrm{Holo}} \geq 1 - \epsilon$ for all $T \leq \mathrm{poly}(d_L)$, which together constitute the Triadic–Fourfold IQR threshold statement.

14.1 Example: Triadic–Fourfold analysis of the standard 9–qubit code

To illustrate the Triadic–Fourfold threshold framework in a concrete and finite setting, we now specialise to the standard 9–qubit Shor code [18, 13, 6], regarded as a single–patch code on the emergent register. This toy example does not realise an asymptotic threshold (since the distance is fixed at $d_L = 3$), but it cleanly exhibits how the triadic error rates, leakage parameters, and process differentials enter the logical and process valuations in a familiar code.

Emergent register and Shor code as a single patch

Let $\mathcal{P}_3 = \{x_1, \ldots, x_9\} \subset \mathfrak{B}$ be a PQS patch of nine loci, and let

$$\mathcal{H}^{(3)}_{\mathrm{reg}} \;:=\; \bigotimes_{j=1}^{9} \mathcal{H}_{(0,1),x_j} \;\cong\; (\mathbb{C}^2)^{\otimes 9}$$

be the emergent register of Definition 11.1 restricted to these loci. We realise the 9–qubit Shor code as a stabilizer code $\mathcal{C}_3 \subset \mathcal{H}^{(3)}_{\mathrm{reg}}$ with distance $d_3 = 3$, using the usual 9–qubit stabilizer group $S_3 \subset \mathcal{P}_9$ and projector

$$P_{\mathrm{Shor}} \;=\; \frac{1}{|S_3|} \sum_{g \in S_3} g.$$

This is a special case of the general stabilizer set–up in Definition 13.1, now with a single code in the family and fixed distance $d_3 = 3$.

The ambient IQR/PQS data $(\mathscr{D}_{\mathrm{QEC}}, \mathcal{I}^{[\mathscr{I}]})$ and Good–clock $C = (\hat{T}, U_t)$ are assumed fixed, as in Assumptions 9.2 and 9.16, so that the emergent register and complement projectors $P^{(3)}_{\mathrm{reg}}, P^{(3)}_{\mathrm{comp}}$ and the triad projectors $\Pi^{(\mathrm{spec})}, \Pi^{(\mathrm{vac})}, \Pi^{(\mathrm{corr})}$ are all well–defined.

A simple triadic noise model on nine sites

We now specify a concrete, but still schematic, Triadic–Fourfold noise model for one QEC cycle on the 9–qubit register. Let $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$ be the corresponding noise kernel, and let $\mathcal{N} : \mathcal{B}(\mathcal{H}_{\mathrm{phys}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ be the full one–cycle channel.

Assumption 14.3 (Toy triadic noise for the 9–qubit code)**.** *We assume:*

(a) Correlational sector. *On each site x_j the correlational contribution induces an approximately independent local depolarising channel on the Twofold fibre,*

$$\mathcal{N}^{(1)}_{j,\mathrm{corr}}(\rho) \;\approx\; (1 - p_{\mathrm{corr}})\,\rho + \frac{p_{\mathrm{corr}}}{3}\Big(X\rho X + Y\rho Y + Z\rho Z\Big), \tag{755}$$

for some $0 < p_{\mathrm{corr}} \ll 1$, common to all sites $j = 1, \ldots, 9$. This is the dominant short–range Pauli–type noise generated by the correlational block $\Delta\hat{G}^{(\mathrm{corr})}_{(0,1)\to(0,1)}$, as in Definition 12.2 and Remark 12.4.

(b) Vacuum sector. *The vacuum contribution acts as a quasi–static, patch–homogeneous Z–rotation during one cycle, with slowly drifting angle,*

$$\mathcal{N}^{(1)}_{j,\mathrm{vac}}(\rho) \;\approx\; e^{-i\theta_{\mathrm{vac}} Z/2}\,\rho\, e^{+i\theta_{\mathrm{vac}} Z/2}, \qquad |\theta_{\mathrm{vac}}| \ll 1, \tag{756}$$

with θ_{vac} drawn from a narrow distribution whose mean and variance vary slowly on time scales $\gg \Delta t_{\mathrm{QEC}}$. The corresponding triadic parameters satisfy $p^{(j)}_{\mathrm{vac}} \sim O(\theta^2_{\mathrm{vac}})$ and, by translational symmetry over the patch, $p^{(j)}_{\mathrm{vac}}$ is independent of j.

(c) Spectral sector. *The spectral sector generates a small probability* $p_{\rm spec} \ll 1$ *of off–band excitation relative to the UV/IR filters and triad projectors, for example via weak coupling to parasymmetric/holo* (p, s) *modes, together with an approximately homogeneous small distortion of the Twofold dynamics. At the single–site level we thus have a channel* $\mathcal{N}^{(1)}_{j,\rm spec}$ *with* $p^{(j)}_{\rm spec} \sim O(p_{\rm spec})$*, again independent of* j *to leading order.*

(d) Triadic time window and clustering. *The triadic clustering Assumption 10.9 and the time–window condition of Definition 10.10 hold, with* $t_{\rm corr} \ll \Delta t_{\rm QEC} \ll t_{\rm vac}, t_{\rm spec}$ *and triadic lengths* ξ_a *large compared to the size of the 9–site patch. Thus the vacuum and spectral contributions are effectively homogeneous on the QEC time and length scales of this example.*

Under this assumption, the single–site triad–split error rates of Definition 12.2 become

$$p^{(j)}_{\rm corr} = p_{\rm corr}, \qquad p^{(j)}_{\rm vac} \sim O(\theta^2_{\rm vac}), \qquad p^{(j)}_{\rm spec} \sim O(p_{\rm spec}), \tag{757}$$

for all $j = 1, \ldots, 9$, and hence the global rates are

$$p_{\rm corr} = p_{\rm corr}, \qquad p_{\rm vac} \sim O(\theta^2_{\rm vac}), \qquad p_{\rm spec} \sim O(p_{\rm spec}). \tag{758}$$

By Lemma 12.3, there exist single–qubit Pauli channels Λ_j approximating the reduced channels $\mathcal{N}^{(1)}_j$, with diamond–norm error

$$\left\| \mathcal{N}^{(1)}_j - \Lambda_j \right\|_\diamond \;\le\; f_{\rm loc}\Big(p_{\rm corr}, p_{\rm vac}, p_{\rm spec}; \{\xi_a\}, \{t_a\}\Big),$$

and, in this symmetric toy model, all nine sites share the same Pauli distribution $\mathbf{q}^{(j)} \equiv \mathbf{q}$.

Logical valuation for the 9–qubit code

For the 9–qubit Shor code under independent local Pauli noise with total physical error rate per site $p_{\rm tot} = p_{\rm corr} + p_{\rm vac} + p_{\rm spec}$, the exact logical error probability $p^{(3)}_{\rm log}$ can be computed combinatorially [18, 13, 6]. In the small–noise regime $p_{\rm tot} \ll 1$ one has the scaling

$$p^{(3)}_{\rm log} \;=\; O\Big(p^2_{\rm tot}\Big), \tag{759}$$

reflecting the fact that a distance–3 code corrects all single–qubit errors but can fail when two or more physical errors occur.

In the triadic setting, Proposition 13.3 specialised to $d_L = 3$ yields a bound of the form

$$p^{(3)}_{\rm log} \;\le\; C\left(\frac{p_{\rm corr}}{p_{\rm corr,*}}\right)^{3c} + f_{\rm vac}\Big(3; p_{\rm vac}, \xi_{\rm vac}, t_{\rm vac}\Big) + f_{\rm spec}\Big(3; p_{\rm spec}, \xi_{\rm spec}, t_{\rm spec}\Big) + O\Big(\varepsilon^{(\rm corr)}_{\rm ps}\Big), \tag{760}$$

where $p_{\rm corr,*}$ plays the role of the reference local–noise threshold value (e.g. inherited from a surface–code family) and $\varepsilon^{(\rm corr)}_{\rm ps}$ is the correlational leakage parameter from Definition 11.4.

In the toy model of Assumption 14.3, the vacuum and spectral sectors are homogeneous on the patch and slowly varying in Good–clock time, so their contributions can be made very small by choosing $|\theta_{\rm vac}| \ll 1$ and $p_{\rm spec} \ll 1$, with coherence scales $(\xi_{\rm vac}, \xi_{\rm spec}; t_{\rm vac}, t_{\rm spec})$ large on the QEC scales. The logical Thesis valuation for the 9–qubit code is then

$$v_{\rm Th}(\rho_{\rm out}; \varphi^{(3)}_3) \;=\; 1 - p^{(3)}_{\rm log} \;\gtrsim\; 1 - C\Big(\frac{p_{\rm corr}}{p_{\rm corr,*}}\Big)^{3c} - f_{\rm vac} - f_{\rm spec} - O\Big(\varepsilon^{(\rm corr)}_{\rm ps}\Big), \tag{761}$$

which concretely realises the general bound (708) in a familiar finite–distance code.

Process holosymmetry for two 9–qubit schedules

Finally, consider two 9–qubit Shor syndrome–extraction schedules A and B (see e.g. [6, 5]), implemented on the same QEC diagnostic realized reference configuration $\mathscr{D}_{\mathrm{QEC}}$ and QEC–realized weave $\mathcal{I}^{[\mathscr{S}]}$, and governed by a common Triadic–Fourfold noise kernel $\Delta\hat{G} \in \mathcal{N}_{\text{tri-4F}}$.

Let $\mathbf{q}^{(j,A)}, \mathbf{q}^{(j,B)}$ be the per–site Pauli distributions and $k^{(j)}_{\mathrm{tot}}, \delta^{(j)}_{\mathrm{tot}}$ the corresponding Bhattacharyya overlaps and process differentials as in Definitions 12.5 and 12.6. In the symmetric toy model we may assume

$$m_L = 9, \qquad \delta^{(j)}_{\mathrm{corr}} = \delta_{\mathrm{corr}}, \qquad \delta^{(j)}_{\mathrm{vac}} \ll \delta_{\mathrm{corr}}, \qquad \delta^{(j)}_{\mathrm{spec}} \ll \delta_{\mathrm{corr}},$$

so that, to leading order,

$$\delta_{\mathrm{tot}} \;\approx\; \delta_{\mathrm{corr}} + \delta_{\mathrm{vac}} + \delta_{\mathrm{spec}} \;\approx\; \delta_{\mathrm{corr}}, \tag{762}$$

in the sense of Remark 12.7. The vacuum and spectral process differentials are small because schedules A and B share the same Good–clock, band–limiting and triad projectors, and thus see essentially the same vacuum and spectral backgrounds, as discussed in Remark 13.9.

For one QEC cycle, Proposition 13.5(a) gives

$$1 - v^{(3)}_{\mathrm{Holo}} \;\le\; \frac{1}{2}\, m_L\, \delta_{\mathrm{tot}} \;=\; \frac{9}{2}\, \delta_{\mathrm{tot}}, \tag{763}$$

and for T cycles,

$$1 - v^{(3,T)}_{\mathrm{Holo}} \;\lesssim\; \frac{1}{2}\, m_L\, T\, \delta_{\mathrm{tot}} \;=\; \frac{9}{2}\, T\, \delta_{\mathrm{tot}}. \tag{764}$$

Imposing the triadic process threshold condition of Definition 13.8 with prescribed accuracy $\epsilon > 0$,

$$\delta_{\mathrm{tot}} \;\le\; \frac{2\epsilon}{m_L T} \;=\; \frac{2\epsilon}{9T}, \tag{765}$$

ensures that

$$v^{(3,T)}_{\mathrm{Holo}} \;\gtrsim\; 1 - \epsilon, \tag{766}$$

for all circuits of length T in the range considered. In particular, for fixed T and sufficiently small δ_{corr} (hence δ_{tot}), the process–equivalence proposition $\varphi^{(3)}_{\mathrm{proc}}$ remains in the Thesis corner of the Tetralemma with error at most ϵ, in line with the general statement of Theorem 14.1.

Remark 14.4 (Role of the 9–qubit example)**.** This example serves as a finite–distance test case for the triadic dictionary: the fourfold inter–modal correlator, triadic clustering scales, and triad–split error, leakage, and process parameters all reduce to explicit quantities for a familiar small code. In the asymptotic surface–code setting of Theorem 14.1, the same mechanisms operate with $d_L \to \infty$ and $|\mathcal{P}_L| \to \infty$, yielding true threshold behaviour; the 9–qubit Shor code can be viewed as the $d_L = 3$ “toy model” that already exhibits the structure of the Triadic–Fourfold threshold conditions in a concrete, calculable case.

Numerical toy instance

To make the previous example fully concrete, we now pick explicit (schematic) values for the triadic parameters and propagate them through the logical and process bounds. The aim is purely illustrative: the constants and thresholds chosen here are not derived from any particular hardware model, but they show how the Triadic–Fourfold framework evaluates a small code in the small–noise regime.

Numerical choice of parameters. We specialise the toy triadic noise of Assumption 14.3 as follows:

$$p_{\text{corr}} = 10^{-3}, \qquad p_{\text{vac}} \sim 10^{-4}, \qquad p_{\text{spec}} \sim 10^{-4}, \tag{767}$$

corresponding, for instance, to $|\theta_{\text{vac}}| \sim 10^{-2}$ and $p_{\text{spec}} \sim 10^{-4}$, and to large triadic coherence scales $(\xi_{\text{vac}}, \xi_{\text{spec}}; t_{\text{vac}}, t_{\text{spec}})$ so that vacuum and spectral noise are slow and homogeneous on the QEC scales of the 9–qubit patch.

For the logical bound, we take schematic threshold and combinatorial constants

$$p_{\text{corr},*} = 10^{-2}, \qquad C = 1, \qquad c = 1, \tag{768}$$

so that $p_{\text{corr}}/p_{\text{corr},*} = 10^{-3}/10^{-2} = 10^{-1}$. We also assume the triadic correction functions and correlational leakage are small enough that

$$f_{\text{vac}} \leq 10^{-4}, \qquad f_{\text{spec}} \leq 10^{-4}, \qquad O\Big(\varepsilon_{\text{ps}}^{(\text{corr})}\Big) \leq 10^{-4}, \tag{769}$$

which is consistent with the qualitative behaviour of $f_{\text{vac}}, f_{\text{spec}}$ and the small–leakage regime described in Proposition 13.3.

For the process side, we pick:

$$m_L = 9, \qquad T = 10^3, \qquad \epsilon = 10^{-2}, \tag{770}$$

so that we are looking at $T = 1000$ QEC cycles with the 9–qubit code, and we demand process–equivalence fidelity at the 1% level.

Logical valuation. The triadic logical error bound for $d_L = 3$ reads

$$p_{\log}^{(3)} \;\leq\; C \left(\frac{p_{\text{corr}}}{p_{\text{corr},*}}\right)^{c\, d_L} + f_{\text{vac}} + f_{\text{spec}} + O\Big(\varepsilon_{\text{ps}}^{(\text{corr})}\Big). \tag{771}$$

With the numerical choices above,

$$C \left(\frac{p_{\text{corr}}}{p_{\text{corr},*}}\right)^{c\, d_L} = 1 \cdot (10^{-1})^3 \;=\; 10^{-3}, \tag{772}$$

$$f_{\text{vac}} + f_{\text{spec}} + O\Big(\varepsilon_{\text{ps}}^{(\text{corr})}\Big) \leq 10^{-4} + 10^{-4} + 10^{-4} \;=\; 3 \times 10^{-4}. \tag{773}$$

Hence

$$p_{\log}^{(3)} \;\leq\; 10^{-3} + 3 \times 10^{-4} \;=\; 1.3 \times 10^{-3}. \tag{774}$$

The corresponding Thesis valuation for the logical correctness proposition $\varphi_3^{(3)}$ is

$$v_{\mathrm{Th}}(\rho_{\mathrm{out}}; \varphi_3^{(3)}) \;=\; 1 - p_{\mathrm{log}}^{(3)} \;\geq\; 1 - 1.3 \times 10^{-3} \;=\; 0.9987. \tag{775}$$

Thus, in this concrete triadic toy model, the 9–qubit code achieves logical correctness at the level of $\sim 0.13\%$ error per cycle, with the dominant contribution coming from the correlational sector, and vacuum/spectral and leakage effects entering as subleading triadic corrections.

Process holosymmetry over $T = 10^3$ cycles. The triadic process holosymmetry bound for T cycles gives

$$1 - v_{\mathrm{Holo}}^{(3,T)} \;\lesssim\; \frac{1}{2}\, m_L\, T\, \delta_{\mathrm{tot}}, \tag{776}$$

where δ_{tot} is the global process differential. The triadic process threshold condition at accuracy ϵ reads

$$\delta_{\mathrm{tot}} \;\leq\; \frac{2\epsilon}{m_L T}. \tag{777}$$

For $m_L = 9$, $T = 10^3$, and $\epsilon = 10^{-2}$, this becomes

$$\delta_{\mathrm{tot}} \;\leq\; \frac{2 \times 10^{-2}}{9 \times 10^3} \;=\; \frac{1}{450000} \;\approx\; 2.22 \times 10^{-6}. \tag{778}$$

Assuming the triadic split $\delta_{\mathrm{tot}} \approx \delta_{\mathrm{corr}} + \delta_{\mathrm{vac}} + \delta_{\mathrm{spec}}$ and that $\delta_{\mathrm{vac}}, \delta_{\mathrm{spec}} \ll \delta_{\mathrm{corr}}$ (as expected when both schedules share the same Good–clock and band–limiting), we may, for instance, choose

$$\delta_{\mathrm{corr}} \approx 2 \times 10^{-6}, \qquad \delta_{\mathrm{vac}}, \delta_{\mathrm{spec}} \lll 10^{-6}, \tag{779}$$

so that $\delta_{\mathrm{tot}} \approx 2.22 \times 10^{-6}$ satisfies the threshold condition.

Substituting $\delta_{\mathrm{tot}} = 1/450000$ into the holosymmetry bound yields

$$1 - v_{\mathrm{Holo}}^{(3,T)} \;\lesssim\; \frac{1}{2}\, m_L\, T\, \delta_{\mathrm{tot}} \;=\; \frac{1}{2} \cdot 9 \cdot 10^3 \cdot \frac{1}{450000} \tag{780}$$

$$= \; \frac{4500}{450000} \;=\; 10^{-2}, \tag{781}$$

so that

$$v_{\mathrm{Holo}}^{(3,T)} \;\gtrsim\; 1 - 10^{-2} \;=\; 0.99, \tag{782}$$

for $T = 10^3$ cycles. In other words, provided the correlational part of the process differential is suppressed to the level of a few parts in 10^6 per cycle, the dual–history holosymmetry between two 9–qubit schedules remains in the Thesis corner with at most $\sim 1\%$ deviation over 10^3 cycles.

Triadic–Fourfold Performance on 9-Qubit Code

With physical noise $p_{\mathrm{corr}} = 10^{-3}$ and backgrounds $p_{\mathrm{vac}}, p_{\mathrm{spec}} \sim 10^{-4}$, the model achieves:

$$\begin{aligned} \mathbf{p}_{\log}^{(\mathbf{3})} &\lesssim \mathbf{1.3 \times 10^{-3}} && \text{(0.13\% logical error / cycle)} \\ \mathbf{v}_{\mathrm{Holo}}^{(\mathbf{3,10^3})} &\gtrsim \mathbf{0.99} && \text{($\sim$1\% process loss over 10^3 cycles)} \end{aligned}$$

Conclusion: Vacuum and spectral "junk" enter only as subleading corrections; the architecture remains robust even in structured, field-theoretic noise.

Remark 14.5 (Numerical toy instance vs. asymptotic threshold)**.** This numerical example should be read as a finite–distance caricature of the Triadic–Fourfold threshold theorem: the 9–qubit code has fixed distance $d_3 = 3$ and thus cannot exhibit a true asymptotic threshold, but it already shows explicitly how:

(i) the correlational error rate p_{corr} dominates the logical error contribution;

(ii) the vacuum and spectral rates $p_{\mathrm{vac}}, p_{\mathrm{spec}}$ and the correlational leakage $\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})}$ enter as controlled triadic corrections; and

(iii) the process differential δ_{tot} controls the holosymmetry valuation over many cycles.

In the full surface–code families of Theorem 14.1, one lets $d_L \to \infty$ while keeping the triadic parameters below their critical values; the same algebraic structure as in this toy instance then yields genuine threshold behaviour with exponentially small logical error and stable process holosymmetry on polynomial–length logical circuits.

In the triadic–Fourfold setting, the surface code itself is not replaced by a new code family. The underlying stabilizer architecture remains the standard one: a planar/surface–code family $\{\mathcal{C}_L\}$ on an emergent register $\mathcal{H}_{\mathrm{reg}}^{(L)}$, with distance $d_L \sim L$ and the usual local–noise threshold p_* for local noise models on stabilizer codes [1, 2, 3, 4, 5]. The novelty lies instead in how the physical conditions under which this *same* code actually functions as a fault–tolerant quantum computer are characterised. Rather than postulating an abstract, flat, Markovian Pauli channel on a rigid lattice, the triadic–Fourfold framework in the IQR setting [8] provides a sharper, more physically grounded control of when a given surface–code architecture remains reliable inside a curved, slowly drifting, non–Markovian environment, in line with general open–system treatments [9, 10].

Standard surface–code threshold analyses effectively compress the noise into a small set of scalar parameters, most prominently a per–site error rate p (sometimes supplemented by a single leakage parameter), within an abstract CPTP or Pauli–channel model [2, 3, 4, 5]. In contrast, the triadic–Fourfold approach resolves the physical noise seen by the emergent Twofold register into triad–split quantities

$$p_{\mathrm{corr}}, \quad p_{\mathrm{vac}}, \quad p_{\mathrm{spec}}, \qquad \varepsilon_{\mathrm{ps}}^{(\mathrm{corr})}, \quad \varepsilon_{\mathrm{ps}}^{(\mathrm{vac})}, \quad \varepsilon_{\mathrm{ps}}^{(\mathrm{spec})},$$

separating fast correlational "junk" from slow background drift and band–structure effects, as encoded in the Triadic–Fourfold Noise Class [8]. The parameters p_{corr} and $\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})}$ quantify the genuinely dangerous, stochastic, Pauli–like component of the noise, and directly control the exponential term in the logical error bound and the true threshold, in close analogy with the role of the physical error rate in standard threshold theorems [1, 2, 3, 4, 5]. By contrast, $p_{\mathrm{vac}}, p_{\mathrm{spec}}$ and their associated leakages enter only through additive correction terms $f_{\mathrm{vac}}, f_{\mathrm{spec}}$ that can be made small by an appropriate choice of QEC time window and patch geometry, reflecting slowly varying vacuum and spectral backgrounds [9, 10, 8]. In place of the usual statement "if the total physical error rate p is below threshold, the code functions fault–tolerantly," the triadic theorem asserts that it suffices for the *correlational* contribution to lie below threshold, and then explicitly quantifies how much slow vacuum and spectral "junk" can accumulate before it begins to affect logical performance. This distinction is particularly important in chronogeometric or gravitational settings, where backgrounds are never exactly static but are often benign on QEC scales [8].

The framework is also explicitly compatible with curved and relational chronogeometries, in a way that standard threshold proofs are not. Conventional analyses assume an external, flat time parameter and a static hardware lattice [1, 2, 3, 4, 5]. In the triadic–Fourfold/IQR setting, all QEC dynamics are anchored in a Good–clock $C = (\hat{T}, U_t)$ and an emergent chronogeometric background [8], with triadic time scales $(t_{\mathrm{corr}}, t_{\mathrm{vac}}, t_{\mathrm{spec}})$ and corresponding lengths $\{\xi_a\}$ governing locality, clustering, and the QEC time window, in the spirit of open–system correlation lengths and times [9, 10]. The QEC patch $\mathcal{P}_L$ is realised as a QEC diagnostic realized reference configuration embedded in a possibly curved or fluctuating background [8]. Chronogeometric drift, background–field variation, and band–structure shear are not swept into an unmodelled "non–Markovian noise" remainder; instead they are systematically packaged into the vacuum and spectral triads and appear as controlled correction terms $f_{\mathrm{vac}}, f_{\mathrm{spec}}$ in the logical error bound. The resulting threshold statement is therefore a surface–code theorem that already "knows" it is living inside an IQR/gravitational field–theoretic context [8].

A further conceptual gain is that the threshold is formulated directly in terms of field–theoretic correlators, rather than phenomenological quantum channels. Standard surface–code theorems typically begin by assuming an i.i.d. Pauli channel or a generic CPTP map acting locally on qubits [2, 3, 4, 5]. In the triadic–Fourfold setting the starting point is a full fourfold inter–modal correlator $\hat{G}_\Lambda$ and its ideal counterpart $\hat{G}_\Lambda^0$ on the underlying field or many–body system [8]. The noise kernel $\Delta\hat{G} := \hat{G}_\Lambda - \hat{G}_\Lambda^0$ is decomposed into triad–resolved blocks and used to define triad–split error rates p_a, triad–resolved leakages $\varepsilon_{\mathrm{ps}}^{(a)}$, and process differentials δ_a as explicit functionals of $\Delta\hat{G}$. The threshold theorem then reads: if the underlying field–theoretic model has correlator $\hat{G}_\Lambda$ in the Triadic–Fourfold Noise Class with sufficiently small triadic parameters, a surface code on the emergent register satisfies the usual fault–tolerance properties [1, 2, 3, 4, 5, 8]. This yields a concrete bridge from microscopic correlator data to macroscopic FTQC guarantees, avoiding an *ad hoc* insertion of Pauli noise by fiat.

Beyond logical correctness, the triadic theorem introduces process–level robustness as a second threshold axis via holosymmetry. Conventional results assert that the logical output of a chosen fault–tolerant implementation is correct with probability at least $1 - \epsilon$ [1, 2, 3, 4, 5].

The triadic analysis retains this but also imposes a bound on the global process differential δ_{tot} between any two QEC schedules implementing the same logical circuit, in the dual–history holosymmetry framework [8]. Under the condition $\delta_{\mathrm{tot}} \lesssim 2\epsilon/(m_L T)$, the dual–history holosymmetric valuation satisfies $v_{\mathrm{Holo}}^{(L,T)} \geq 1 - \epsilon$ for all circuits of length T, meaning that all such schedules are physically equivalent in the dual–history sense up to accuracy ϵ [8]. Fault tolerance is therefore not only a statement about the correctness of a single compiled schedule, but also a guarantee that reasonable variations in scheduling and compilation do not appreciably change the realised physical process, provided they do not significantly increase δ_{tot}. This process–equivalence axis is typically absent from standard threshold formulations [2, 3, 5].

Leakage is treated with corresponding refinement. Instead of a single scalar leakage rate, the analysis employs a basic leakage norm $\varepsilon_{\mathrm{ps}}$ together with triad–resolved components $\varepsilon_{\mathrm{ps}}^{(\mathrm{corr})}, \varepsilon_{\mathrm{ps}}^{(\mathrm{vac})}, \varepsilon_{\mathrm{ps}}^{(\mathrm{spec})}$, in harmony with leakage/error parameters used in FTQC [17, 3, 5, 8]. A comparison lemma shows that $\varepsilon_{\mathrm{ps}}$ is controlled by these pieces, with the correlational component singled out as the dominant contribution in the threshold bound, while the vacuum and spectral leakage terms can be absorbed into the slowly varying correction functions. This allows a clear distinction between fast Twofold $\leftrightarrow (p, s)$ mixing, which must be strongly suppressed at the hardware and control level, and slow leakage mechanisms that behave more like background dressing of the effective Hamiltonian and can be tolerated within the triadic QEC time window [9, 10, 8].

Finally, the entire result is formulated in terms of Tetralemma valuations for both logical and process propositions, as developed in the IQR framework [8]. Logical correctness $\varphi_L^{(L)}$ and process equivalence $\varphi_{\mathrm{proc}}^{(L)}$ are assigned Thesis valuations v_{Th} and v_{Holo}, and the threshold theorem shows that both can be kept in the Thesis corner with error at most ϵ once the triadic inequalities are satisfied. The familiar statement "the logical error rate is $\leq \epsilon$" is thus integrated into a wider logical and interpretive structure that applies not only to final outcomes but also to entire computational histories and schedule classes. In practical terms, the architecture remains that of the standard surface code on an emergent register, and the dominant threshold condition remains $p_{\mathrm{corr}} < p_{\mathrm{corr},*}$. The advantage of the triadic–Fourfold approach is that this familiar architecture is now justified within a strictly more general and physically faithful noise class: one that accommodates curved chronogeometries, slow vacuum and spectral drift, structured leakage, and schedule–level variations, while still delivering the exponential suppression of logical and process errors characteristic of fault–tolerant surface–code quantum computation [1, 2, 3, 4, 5, 8].

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